ISSN 0019-5588

Indian Journal of Pure & Applied Mathematics

DEVOTED PRIMARILY TO ORIGINAL RESEARCH IN PURE AND APPLIED MATHEMATICS

INDIAN NATIONAL SCIENCE ACADEMY

हव्यामि भर्गः सवितुर्वरेण्यम्

VOLUME 20/8
AUGUST 1989

INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS

Published monthly by the

INDIAN NATIONAL SCIENCE ACADEMY

Subscriptions :

For India, Pakistan, Sri Lanka, Nepal, Bangladesh and Burma, Contact :

Associate Editor, Indian National Science Academy, Bahadur Shah Zafar Marg, New Delhi 110002, Telephone : 3311865, Telex : 31-61835 INSA IN.

For other countries, Contact :

M/s J. C. Baltzer AG, Scientific Publishing Company, Wettsteinplatz 10, CH-4058 Basel, Switzerland, Telephone : 61-268925, Telex : 63475.

The Journal is indexed in the Science Citation Index; Current Contents (Physical, Chemical & Earth Sciences); Mathematical Reviews; INSPEC Science Abstracts (Part A); as well as all the major abstracting services of the World.

Indian J. pure appl. Math., **20** (8) : 747–754, August 1989

COSET DIAGRAMS FOR AN ACTION OF THE EXTENDED MODULAR GROUP ON THE PROJECTIVE LINE OVER A FINITE FIELD

QAISER MUSHTAQ

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

(*Received* 19 *November* 1987; *after revision* 20 *November* 1988)

Higman has defined coset diagrams for the actions of $PGL\ (2, Z)$ on the projective line over a finite field F_q, denoted by $PL\ (F_q)$, where q is a prime power. A condition for the existence of a certain fragment of a coset diagram in a coset diagram for an action of $PGL\ (2, Z)$ on $PL\ (F_q)$ is known now: the condition is a polynomial $Z\ [z]$. There are special types of fragments of coset diagrams which occur quite frequently in certain coset diagrams. In this paper we have found conditions for their existence in coset diagrams representing these actions.

1. INTRODUCTION

If q is a power of a prime p then by $PL\ (F_q)$ we shall mean the projective line over the Galois field F_q. That is, $PL\ (F_q) = F_q \cup \{\infty\}$. The group $PGL\ (2, q)$ has its customary meaning, as the group of all transformations $z \to (az + b)/(c z + d)$ where a, b, c, d are in F_q and $ad-bc \neq 0$, while the group $PSL\ (2, q)$ is its subgroup consisting of all those where $ad-bc$ is a non-zero square in F_q. Mushtaq[4] it has shown that corresponding to each θ in F_q there exists a coset diagram which represents an action of the modular group $PSL\ (2, Z) = < x, y : x^2 = y^3 = 1 >$ or the extended modular group $PGL\ (2, Z) = <x, y, t : x^2 = y^3 = t^2 = (x\,t)^2 = (y\,t)^2 = 1 >$ on $PL\ (F_q)$. Consideration of these actions shows the importance of circuits contained in the coset diagrams representing these actions[5]. In this paper we have found values of θ and q, for the occurence of a certain circuit in the corresponding coset diagram.

2. COSET DIAGRAMS FOR $PGL\ (2, Z)$

The coset diagrams considered in this paper are defined for $PGL\ (2, Z)$ (see for details Conder[1 3] and Mushtaq[4,5]). The three cycles of y are denoted by small triangles whose vertices are permuted counter-clockwise by y and any two vertices which are interchanged by x are joined by an edge. The action of t is given by a reflection in a vertical axis of symmetry. The fixed points of x and y are denoted by heavy dots. (Notice $(yt)^2 = 1$ is equivalent to $tyt = y^{-1}$, which means that t reverses the orientation of the triangles representing the three cycles of y (as reflection does); because of this, there is no need to make the diagram more complicated by introducing t-edges.)

The group PGL (2, q) has a natural permutation representation on PL (F_q) and therefore any non-degenerate homomorphism α of PGL (2, Z) to PGL (2, q) gives rise to an action of PGL (2, Z) on PL (F_q). Two such homomorphisms α and β are called conjugate if $\beta = \alpha\rho$ for some inner automorphism ρ of PGL (2, q). We denote $x\alpha$, $y\alpha$ and $t\alpha$ respectively by $\bar{x}, \bar{y}, \bar{t}$. The actions corresponding to conjugate non-degenerate homomorphisms α and β (see Mushtaq[6] for definitions) will produce the same coset diagrams, except for the labelling of the vertices. (The vertices for one can always be re-labelled according to the action of ρ or ρ^{-1} in order to obtain the other).

It has been shown by Mushtaq[4], that the conjugacy classes of non-degenerate homomorphisms of PGL (2, Z) into PGL (2, q) correspond in a one-to-one fashion with the conjugacy classes of non-trivial elements of PGL (2, q), under a correspondence which assigns to the homomorphism α the class containing the element $\bar{x}\,\bar{y}$.

Since it has been proved by Mushtaq[4] that there is a one-to-one correspondence between the conjugacy classes of elements of order greater than 2 in PGL (2, q) and the non-zero elements of F_q, such that the class corresponding to an element θ in F_q consists of all elements represented by matrices M with $\theta = r^2/\Delta$ where r = trace (M) and Δ = det (M), it follows that we can actually parameterize the non-degenerate homomorphisms of PGL (2, Z) into PGL (2, q), exccpt for a few uninteresting ones, by the elements of F_q. If α is any such homomorphism, and X, Y and T are in GL (2, q), which yield the elements $\bar{x}, \bar{y}, \bar{t}$ then letting $\theta = r^2/\Delta$ (where r = trace (XY), Δ = det (XY)), we associate the parameter θ with the homomorphism α.

3. Occurrence of Circuits in D (θ, q)

By a circuit (in a coset diagram for an action of PGL (2, Z) on PL (F_q)) we shall means a closed path of triangles and edges. Coset diagrams arising from non-degenerate actions of PGL (2, Z) on PL (F_q) may be thought of as being composed of fragments, these fragements themselves being composed of single circuits, or a number of circuits[5]. Let us denote by D (θ, q) the diagram for a non-degenerate homomorphism of PGL (2, Z) into PGL (2, Z) with parameter θ. In order to know which class (es) a particular diagram comes from we need to consider this question: given a fragment, for which values of q and θ can that fragment be found in D (θ, q)? The question has been answered by Mushtaq[5] for the fragments containing two or more intersecting circuits. In the following, we have considered the above question for single circuits.

Note that if v is a vertex of a triangle on a circuit in D (θ, q) then the circuit induces an element $\bar{g}$ of PGL (2, q) fixing v. The element $\bar{g} \neq 1$ is conjugate either to $\bar{x}$ or to $\bar{y}^{\pm 1}$ or to an element of the form $\bar{x}\bar{y}^{\epsilon_1}\,\bar{x}\bar{y}^{\epsilon_2} \ldots \bar{x}\bar{y}^{\epsilon_k}$, $k > 1$ and $\epsilon_i = \pm 1$. The expression is unique up to cyclic permutations of ($\epsilon_1, \epsilon_2, \ldots \epsilon_k$). If $\bar{g} = \bar{h}^{k'}$ for some $k' \geqslant 1$ then k' divides k and the cycle consists of ($\epsilon_1, \epsilon_2, \ldots, \epsilon_{k/k'}$) taken k'-times. It means that for any such $\bar{g}$ there is greatest integer k' such that $\bar{g}$ is a k'th power in PGL (2, q). We say that $\bar{g}$ is a proper power if $k' > 1$.

Suppose $D(\theta, q)$ contains circuits which have more than two fixed vertices of the elements conjugate to $\bar{h}$ where $\bar{h}^k \neq 1$. Since the only element of $PGL(2, q)$ with more than two fixed vertices is the identity element, $\bar{g} = 1$. If $\bar{h} = \bar{x}\bar{y}$ the diagram will correspond to an action of $\Delta(2, 3, k) = \langle x, y: x^2 = y^3 = (x\,y)^k = 1 \rangle$ on $PL(F_q)$. In the following theorems, we consider the coset diagrams which correspond to the actions of

$\langle x, y: x^2 = y^3 = xy^{\epsilon_1}\, xy^{\epsilon_2} \ldots xy^{\epsilon_k} = 1, k > 1 \rangle$ on $PL(F_q)$. Mushtaq[4] it has shown that in $D(\theta, q)$, there do not exist circuits which contain fixed points of $\bar{g} = (\bar{h})^k$ where $\bar{h} = (\bar{x}y^{-1})^{n_1} (\bar{x}\bar{y}^{-1})^{n_2} \ldots (\bar{x}\bar{y}^{-1})^{n_k}$, $k > 1$. We call such circuits periodic. A circuit which is not of this type is called a non-periodic circuit.

We are now going to consider the following question. Given a non-periodic circuit, for what values of q, θ can this circuit be found in the corresponding diagram, representing an action of $PGL(2, Z)$ on $PL(F_q)$? We have seen that $\bar{g}$ in $PGL(2, q)$ is either a proper power or it is not. The case where $\bar{g}$ is a proper power has already been considered by Mushtaq[6]. In the following Theorem we therefore deal with the case where $\bar{g}$ is not a proper power. For the meaning of homomorphic image of a circuit and how a non-periodic circuit can have a homomorphic image, we may refer to Mushtaq[5].

Theorem 3.1—Let C be a non-periodic circuit. Then there exists a polynomial f, with integer coefficients, such that if C occurs in $D(\theta, q)$ then $f(\theta)$ is a square in F_q, and if $f(\theta)$ is a square in F_q then either C, or a homomorphic image of it, occurs in $D(\theta, q)$.

PROOF: Let $\bar{g} = \bar{x}\bar{y}^{\epsilon_1}\, \bar{x}\bar{y}^{\epsilon_2} \ldots \bar{x}\bar{y}^{\epsilon_k}$, where $\epsilon_i = \pm 1$ and $i = 1, 2, \ldots, k$, be an element of $PGL(2, q)$ fixing a particular vertex v on the circuit C. The matrices X, Y in $GL(2, q)$, yielding the elements $\bar{x}, \bar{y}$, induce the matrix $M = XY^{\epsilon_1}\, XY^{\epsilon_2} \ldots XY^{\epsilon_k}$ where $\epsilon_i = \pm 1$ and $i = 1, 2, \ldots, k$.

Since $\bar{x}^2 = \bar{y}^3 = 1$, the matrices X and Y can be taken as the matrices with $\det(X) = \Delta$, trace $(X) = 0$ and $\det(Y) = 1$, trace $(Y) = -1$. Thus, as in a paper by Mushtaq[5] characteristic equations of X, XY and Y will be

$$X^2 + \Delta I = 0 \qquad \ldots(1)$$

$$(XY)^2 - r(XY) + \Delta I = 0 \qquad \ldots(2)$$

$$Y^2 + Y + I = 0 \qquad \ldots(3)$$

where the trace of XY is r. Since $\det(XY) = \Delta$, the determinant of M will be Δ^k. Using equation (1), (2) and (3) the matrix M can be expressed as $\lambda_0 1 + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$, where λ_i is a polynomial in r and Δ. So the trace of will be the trace of $M\ \lambda_0 1 + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$. That is, trace $(M) = 2\lambda_0 - \lambda_2 + \lambda_3 r$. So the characteristic equation of M will be

$$M^2 - (2\lambda_0 - \lambda_2 + \lambda_3 r)\, M + \Delta^k I = 0. \qquad \ldots(4)$$

The discriminant $(2\lambda_0 + \lambda_2 + \lambda_3 r)^2 - 4\Delta^k$ of the characteristic equation of M is a polynomial in r and Δ. It can be seen (by induction on k), that if we regard r as of degree 1, and Δ as of degree 2, then this polynomial is homogeneous of degree $2k$. It follows that for a suitable $h(\theta)$, the discriminant is $h(\theta)\Delta^k$.

Now $\bar{g}$ has a fixed vertex in $D(\theta, q)$ if the characteristic equation of M has roots in F_q. This means that $\bar{g}$ has a fixed vertex in $PL(F_q)$, if the discriminant $h(\theta)\Delta^k$ is a square in F_q. Since Δ is a square if and only if θ is, we let $f(\theta) = h(\theta)$ if k is even and $f(\theta) = h(\theta)\theta$ if k is odd and obtain the result in the above theorem.

The preceding theorem has the following interesting corollary.

Corollary 3.2—Let C be circuit such that it contains a vertex which is fixed by $\bar{x}y^{-1}\bar{x}y$. Then C, or a homomorphic image of it occurs in $D(\theta, q)$ if and only if $\theta^2 - 2\theta - 3$ is a square in F_q.

PROOF : Consider the vertex v (on the circuit) fixed by $\bar{x}y^{-1}\bar{x}y$. Let $\bar{g} = \bar{x}y^{-1}\bar{x}y$, then by the direct application of Theorem 3.1, the characteristic equation of the matrix M, corresponding to $\bar{g}$, will be

$$M^2 - (-r^2 + \Delta)M + \Delta I = 0. \qquad \ldots(5)$$

Substituting $r^2 = \theta\Delta$ in the discriminant of (5) we get $f(\theta) = \theta^2 - 2\theta - 3$. Thus the circuit C, or its homomorphic image, exists in $D(\theta, q)$ if and only if $f(\theta)$ is a square in F_q.

Note that the circuit C, which contains a vertex v fixed by $\bar{x}y^{-1}\bar{x}y$, will be as follows

V

FIG. 1.

Remarks 3.3 : (i) The degree of the polynomial $f(\theta)$ will be k or $k+1$, where k is the number of triangles on the circuit C.

(ii) According to Theorem 3.1, the circuit (Fig. 2) occurs in $D(\theta, q)$ if and only if -3 is a square in F_q.

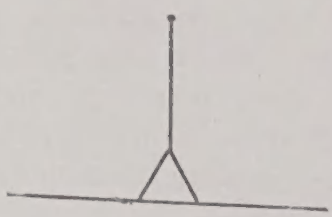

FIG. 2.

(iii) The circuit (Fig. 3) occurs in $D(\theta, q)$ if and only if $-\theta$ is a square in F_q. This is an immediate consequence of Theorem 3.1 and the fact that discriminant in (1) is equal to -4Δ, $r^2 = \Delta\theta$ and

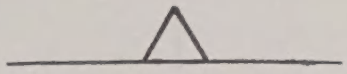

FIG. 3.

(iv) The elements $\bar{x}$, $\bar{t}$ generate a 4-group. In characteristic 2, the only irreducible linear representation of the 4-group is the trivial representation and so in any projective representation there is a fixed vertex. This means $\bar{x}$ and $\bar{t}$ must have a common vertex. Thus a circuit (Fig. 4) falls on the vertical axis of symmetry of $D(\theta, q)$ if and only if q is a power of 2.

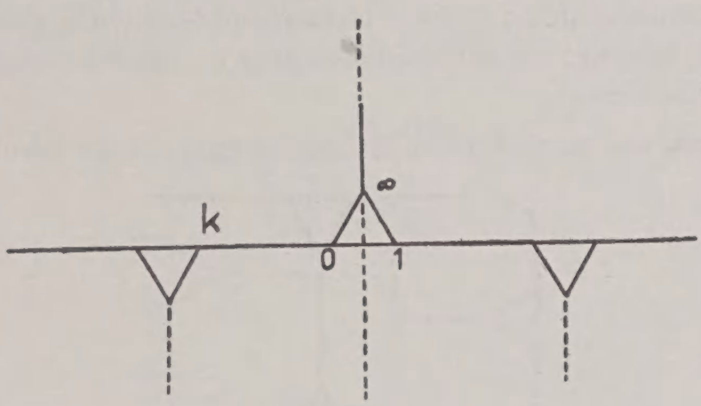

FIG. 4.

(v) If q is a power of 3, then $\bar{y}$ has a unique fixed vertex because the number of vertices in $PL(F_q)$ is $3^k + 1$; $\bar{t}$ normalizes $<\bar{y}>$ and so $\bar{t}$ must also fix this vertex. Thus the circuit (Fig. 5) falls on the vertical axis of symmetery of $D(\theta, q)$.

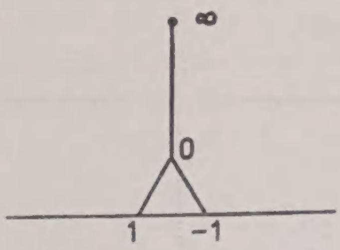

FIG. 5.

Corollary 3.4—A circuit, containing a vertex fixed by $\bar{x}\,\bar{y}$, will exist in $D(\theta, q)$ if and only if $\theta(\theta - 4)$ is a square in F_q.

PROOF : The proof is an easy consequence of Theorem 3.1.

Note that the circuit which contains a fixed point of $\bar{x}\,\bar{y}$ will be (see Fig. 6)

FIG. 6.

4. CIRCUITS WHEN THE DISCRIMINANT IS ZERO

An obvious question which arises here is, what happens when the discriminant of eqn. (4) is equal to zero? Exactly what this means depends upon the circuit.

Let C be a circuit such that a vertex in it is fixed by an element $\bar{g}$ of $PGL\ (2, q)$. The characteristic equation of matrix corresponding to $\bar{g}$ will, of course, have equal eigen values if the discriminant of the characteristic equation is equal to zero. This means that $\bar{g}$ will have just one fixed vertex in $D\,(\theta, q)$, but the exact type will depend upon the circuit concerned.

For instance, assume that the homomorphic image of the circuit (Fig. 7) occurs

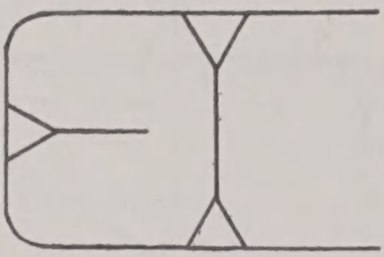

FIG. 7.

in the coset diagram $D\,(\theta, q)$. Since $D\,(\theta, q)$ admits the axis of symmetry, the image of the circuit under the permutation $\bar{t}$, will also occur. The vertices v and $v\,\bar{t}\,\bar{y}^{-1}$ on

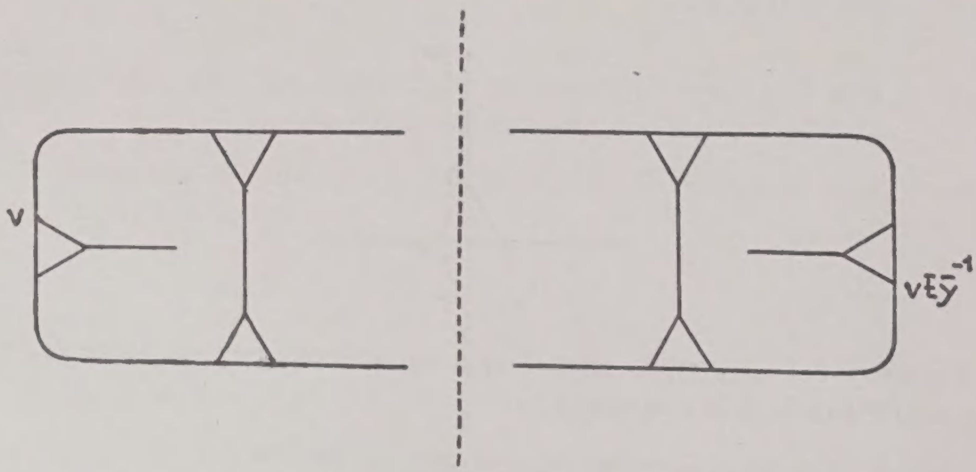

FIG. 8.

the circuits (Fig. 8) are both fixed by $\bar{g}$, where $\bar{g} = \bar{x}\, y\, \bar{x}\, y\, \bar{x}\, y^{-1}$. So if the discriminant of the characteristic equation of the matrix corresponding to $\bar{g}$ is equal to zero then $v = v\, \bar{t}\, y^{-1}$. This means that the circuit, which has a symmatry, lies on the vertical axis of symmetry Diagrammatically it means, (Fig. 9).

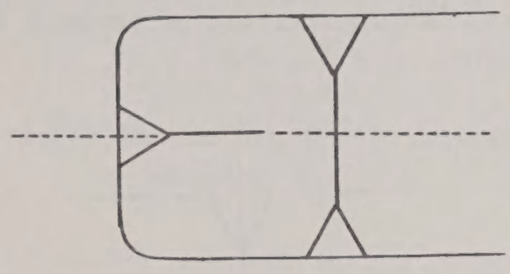

FIG. 9.

Consider another example in which the vertices v_1 and v_2 on the circuit (Fig. 10) are fixed points of $\bar{g}$, where $\bar{g} = \bar{x}\, y\, \bar{x}\, y\, \bar{x}\, y^{-1}\, \bar{x}\, y^{-1}$. In this case when the discriminant of the characteristic equation of the matrix corresponding to $\bar{g}$ is zero we must,

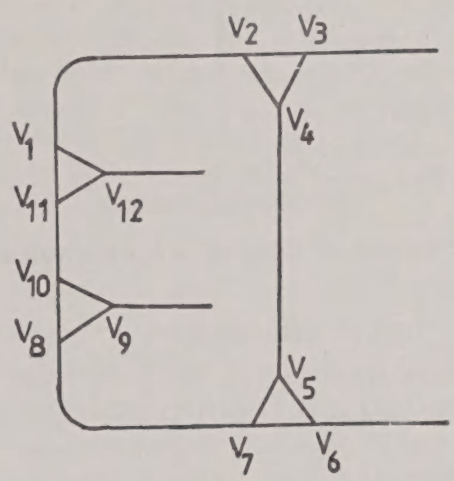

FIG. 10.

get a homomorphic image of the circuit in which v_1 and v_2 are identified, say with $v\, \bar{x} = v$, and so a homomorphic image of the circuit will be (Fig. 11) and it will be symmetrical about the vertical line of axis because $(\bar{x}\, y)^2 = 1$, that is because $v_2 = v_0\, \bar{t}$.

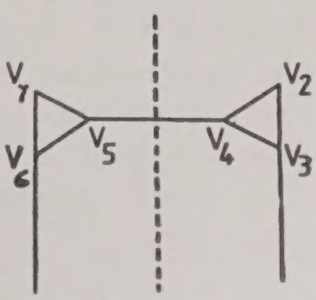

FIG. 11.

If the discriminant $(\theta - 4)\,\Delta$ of the characteristic equation in Corollary 3.4 is equal to zero then $\theta = 4$ because Δ, being the determinant of the non-singular matrix XY, cannot be zero. This implies that the characteristic equation will have equal roots and so there will be only one vertex, namely ∞, in $D(\theta, q)$ fixed by $\bar{x}$. Since the action of t represent reflection about the vertical line of symmetry, the circuit (Fig. 12) will, in this case, lie on the vertical axis of symmetry.

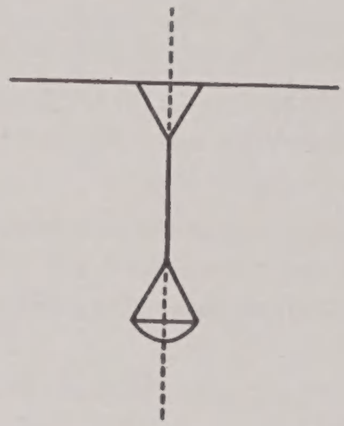

FIG. 12.

ACKNOWLEDGEMENT

I am indebted to the referee of this paper for his valuable suggestions.

REFERENCES

1. M. D. E. Conder, *J. London Math. Soc.* (2) **22** (1980), 75–86.
2. M. D. E. Conder, *Quart. J. Math. Oxford* (2) **32** (1981), 157-63.
3. M. D. E. Conder, *Bull. Austral. Math. Soc.* **30** (1984), 73–90.
4. Q. Mushtaq, D. Phil. Thesis, University of Oxford, 1983.
5. Q. Mushtaq, *Quart. J. Math. Oxford* (2) **39** (1988), 81–95.
6. Q. Mushtaq, *Math. Chronicle* **16** (1987), 69–77.

Indian J. pure appl. Math., **20** (8) : 755–760, August 1989

THE EXTENDED MODULAR GROUP ACTING ON THE PROJECTIVE LINE OVER A GALOIS FIELD

Q. MUSHTAQ

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

(*Received* 19 *November* 1987; *after revision* 20 *November* 1988)

In his Oxford seminars, G. Higman considered actions of $\Delta(2, 3, 7) = < x, y : x^2 = y^3 = (xy)^7 = 1 >$ on the projective line over F_p (denoted by $PL(F_p)$) in the case when p is a prime (congruent to ± 1 mod 7). From the coset diagrams representing the natural action of $\Delta(2, 3, 7)$ on $PL(F_p)$ something of a pattern emerged: for each prime there were three diagrams, but just one of these three had an axis of symmetry passing through two vertices. It was easy to identify the permutation t, induced by the symmetry about this axis, as being odd or even according to the value of $p \equiv \pm 1 \pmod 4$, and then correspondingly, the group $< x, y, t >$ was either $PGL(2, p)$ or just $PSL(2, p)$. In this paper we show that this pattern is not repeated for all values of p.

Let $\Delta(2, 3, 7)$ denote the abstract group with presentation $< x, y : x^2 = y^3 = (xy)^7 = 1 >$. It is well known that $PSL(2, Z)$ has the presentation $< x, y : x^2 = y^8 = 1 >$. Let p be a prime and F_p denote a finite field of order p. We use the notation $GL(2, p)$, $SL(2, p)$, $PSL(2, p)$ and $PGL(2, p)$ with its standard meaning.

Let $PL(F_p)$ denote the projective line over a finite field F_p. The points of $PL(F_p)$ are the elements of F_p together with the additional point ∞. It is well known that the extended modular group $PGL(2, Z)$ has the presentation

$$< x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 >$$

and the modular group $PSL(2, Z)$ is of index 2 in $PGL(2, Z)$. The coset diagrams for the natural action of $PGL(2, Z)$ on $PL(F_p)$ are defined as follows.

The three cycles of y are denoted by triangles whose vertices are permuted anticlockwise by y. Any two vertices which are interchanged by x are joined by an edge. The fixed points of y are denoted by heavy dots and the action of t is given by reflection in a vertical axis of symmetry.

For instance, the following diagram depicts a transitive action of $PGL(2, Z)$ on $PL(F_{13})$, in which

x acts as $(0, 11)(1, 12)(2, 7)(3)(4, 8)(5, 10)(6, \infty)(9)$

y acts as $(0)(\infty)(1, 9, 3)(2, 5, 6)(4, 10, 12)(7, 11, 8)$ and

t acts as $(0, \infty)(2, 7)(3, 9)(4, 10)(5, 8)(1)(12)$.

We have labelled each vertex to give a fuller illustration (Fig. 1).

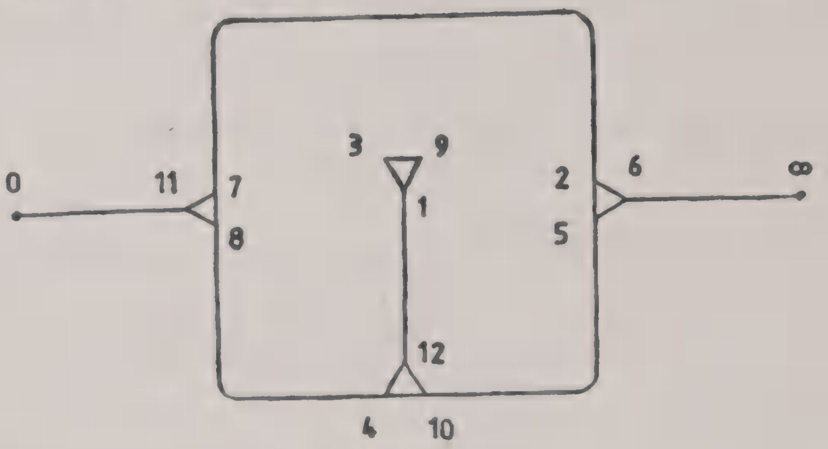

FIG. 1.

Notice that this diagram is symmetric : the permutation $t = (0, \infty)(2, 7)(3, 9)(4, 10)(5, 8)(1)(12)$ gives a reflection about a vertical axis. Also this permutation is the same as that induced by the transformation t which is an element of $PGL\ (2, 13)$ and it is easy to verify geometrically that the relations $t^2 = (xt)^2 = (yt)^2 = 1$ are satisfied.

Indeed in any case, if (x, y) is a $(2, 3, 7)$—generating pair for $PSL\ (2, p)$, then by the proof of Theorem 3 of Singerman[5], there has to be an automorphism σ of $PSL\ (2, p)$ such that $x\sigma = x^{-1}$ and $y\sigma = y^{-1}$. In terms of the associated coset diagram, this means there must be an axis of symmetry, with σ representable as a reflection about the axis. Hence all our diagrams for the groups $PSL\ (2, p)$ will be symmetric. It is important to note that, there are certain coset diagrams (e.g., the coset diagram for $PGL\ (2, 25)$) which admit more than one such symmetry. For details one can refer to Mushtaq[4]. Note that we can form the semi-direct product G of $PSL\ (2, p)$ by the cyclic group $\langle\sigma\rangle$ and in this group the relations $\sigma^2 = (x\sigma)^2 = (y\sigma)^2 = 1$ are satisfied. It is worthy of mention that in Conder[1] has used these coset diagrams to show that all but a small number of alternating groups A_n are homomorphic images of the triangle group $\Delta\ (2, 3, 7)$.

Earlier the author[3] has obtained a parametrization of all homomorphisms α from the extended modular group into $PGL\ (2, q)$ (where q is a prime-power), via non-zero elements of F_q. If neither of the generators x and y for the modular group lies in the kernel of α, so that their images $x\alpha$ and $y\alpha$ are of orders 2 and 3, respectively, then α is said to be a non-degenerate homomorphism. Two such homomorphisms α and β are called conjugute if $\beta = \alpha\rho$ for some inner automorphism ρ of $PGL\ (2, q)$. In this case the actions corresponding to α and β will produce the same coset diagrams, except for the labelling of the vertices. Thus, it has been shown[3] that corresponding to each

θ in F_q there exists a coset diagram $D(\theta, q)$ which represents the conjugacy class of non-degenerate homomorphisms α.

In the present case, since q is taken to be a prime $p \equiv \pm 1 \pmod 7$, according to Macbeath[2], there are three distinct traces $\lambda_1, \lambda_2, \lambda_3$ of elements of the group $SL(2, p)$ that yield elements of order 7 in $PSL(2, p)$ and thus, corresponding to $\theta_1 = \lambda_1^2$, $\theta_2 = \lambda_2^2$ and $\theta_3 = \lambda_3^2$, there are three conjugacy classes of non-degenerate homomorphisms from $\Delta(2, 3, 7)$ into $PGL(2, q)$. This means that there are three coset diagrams $D(\theta_1, p)$, $D(\theta_2, p)$ and $D(\theta_3, p)$ corresponding to the three conjugacy classes[3].

It is important to note that in this case, every element of $PSL(2, p)$ that comes from an element of $SL(2, p)$ with trace λ_1, λ_2 or λ_3 must have order 7. (Indeed except when the trace is ± 2, the trace of any element of $SL(2, p)$ determines its order).

G. Higman, in his Oxford seminars, considered conjugacy classes of non-degenerate homomophisms from $\Delta(2, 3, 7)$ into $PGL(2, p)$ where $p \equiv \pm 1 \pmod 7$. According to Macbeath[2], for each such prime p there are three conjugacy classes. Higman considered coset diagrams corresponding to each of these conjugacy classes. From the coset diagrams he produced in the cases with $p = 13, 29, 41, 43$ and 71, something of a pattern emerged : for each prime there were three diagrams, but just one of these

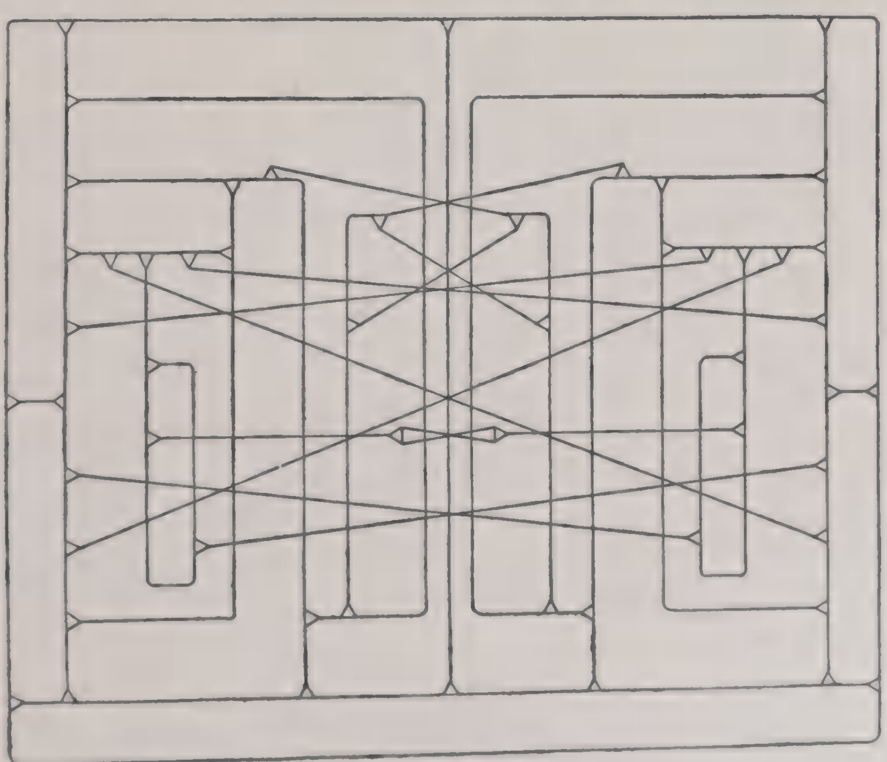

FIG. 2.

three had an axis of symmetry passing through two vertices. In other words, in one diagram the transformation t fixed two points of the projective line over F_p, while in the other two diagrams t fixed no points. It was easy to identify the permutation induced by t as being either odd or even, according to the value of p (mod 4), and then correspondingly, the group $<x, y, t>$ was either $PGL(2, p)$ or just $PSL(2, p)$. Unfortunately this pattern is not repeated for all longer values of p, as we shall see in the following example.

Following the method of Mushtaq[3], we have drawn three coset diagrams $D(\theta_1, p)$, $D(\theta_2, p)$, $D(\theta_3, p)$ for each value of $p = 7, 13, 29, 41, 43, 71, 83, 97, 113, 127, 139$ and 167. We have seen that $p = 167$ is the first case in which the three coset diagrams, namely, $D(21, 167)$, $D(27, 167)$ and $D(124, 167)$ are such that every vertex in them is fixed by $\{(xy)\alpha\}^7$ and the three diagrams contain an axis of symmetry passing through two vertices. Since no non-trivial linear-fractional transformation fixes more than two vertices of $PL(F_q)$, we have $\{(xy)\alpha\}^7 = 1$.

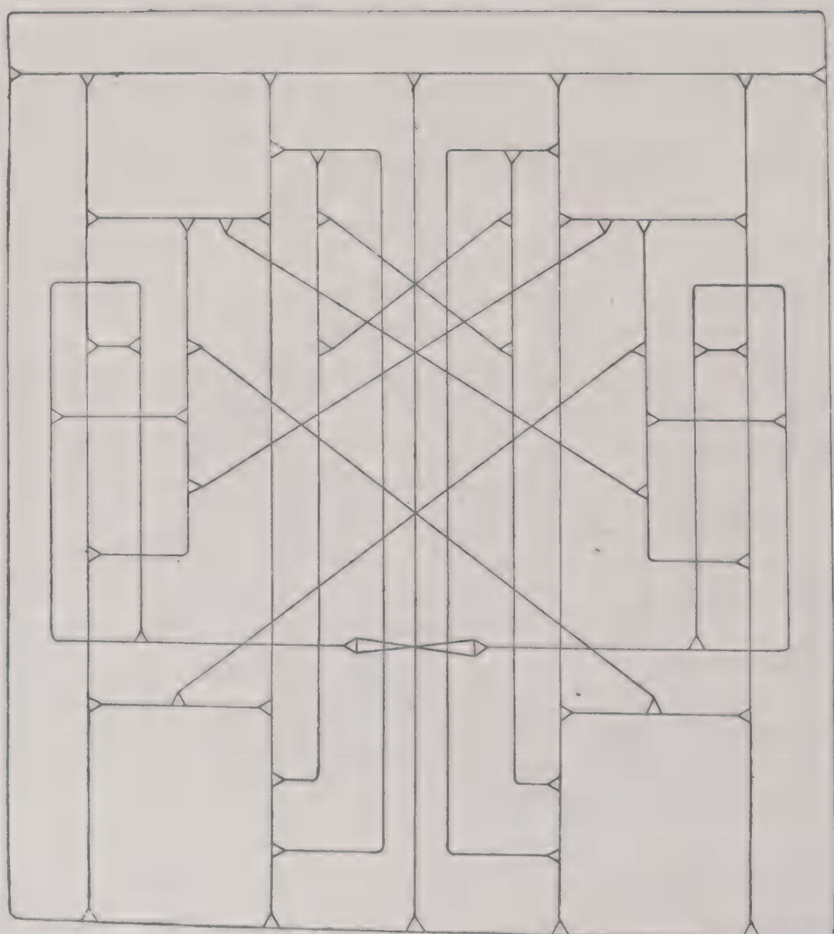

FIG. 3.

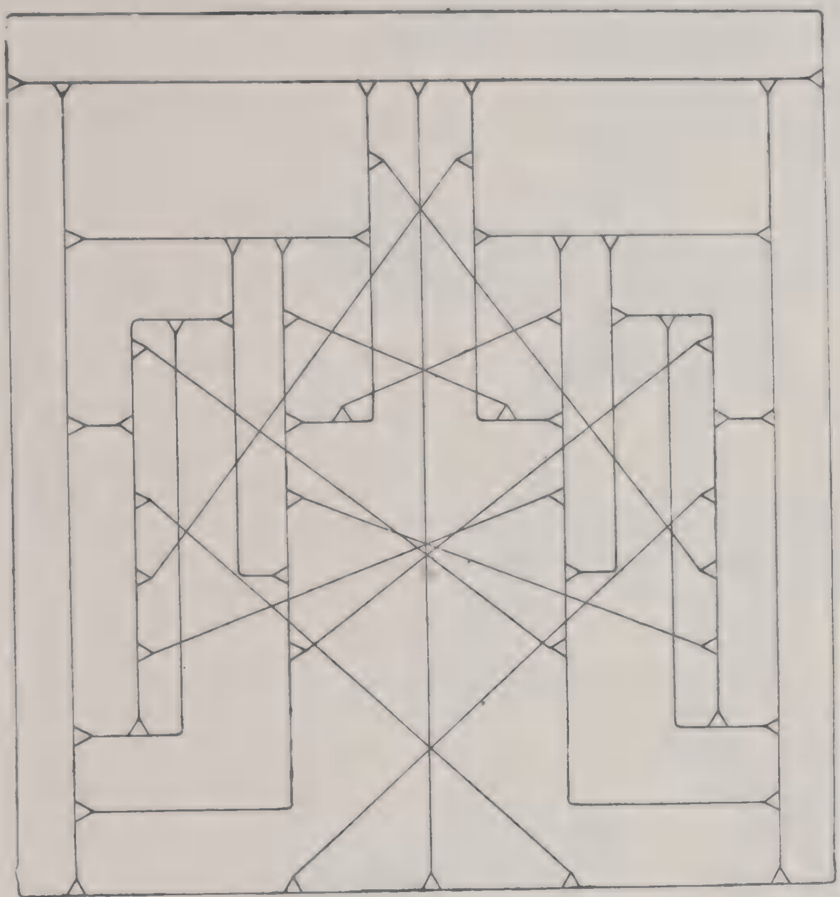

FIG. 4.

In other words, in the three diagrams, $\{(xy)\,\alpha\}^7 = 1$ and the permutation t fixes two points of $PL\,(F_{167})$. It is quite easy to see that in this case t is an odd permutation on $PL\,(F_{167})$ and as a consequence $t \in PGL\,(2, 167) \setminus PSL\,(2, 167)$.

We note that the three equations $\theta - 21 = 0$, $\theta - 27 = 0$ and $\theta - 124 = 0$, when multiplied together, yield the equation $f(\theta) = \theta^3 - 5\theta^2 + 6\theta - 1 = 0$. Notice that this is the same equation as discussed in Mushtaq[3].

ACKNOWLEDGEMENT

Author would like to thank A. Karim and F. Shaheen for their help in drawing the coset diagrams contained in the paper. Thanks are also due to the Quaid-i-Azam University for the research grant during the year 1985-86 which enabled me to carry out this research. Thanks are also due to the referee for his valuable remarks.

REFERENCES

1. M. D. E. Conder, *J. London Math. Soc.* (2), **22** (1980), 75–86.
2. A. M. Macbeath, *Proc. of Symposia in Pure Math.*, 12, *Amer. Math. Soc. Proc.*, (1969), 14–32.

3. Q. Mushtaq, D. Phil. Thesis, 1983, Oxford.
4. Q. Mushtaq, *Math. Chronicle* **16** (1987), 69–77.
5. D. Singerman, *Math. Annalen* **210** (1974), 17–32.

Indian J. pure appl. Math., **20** (8) : 761–767, August 1989

ON THE EXISTENCE FOR A CLASS OF OPTIMAL CONTROL PROBLEMS

GARGI CHAKRABORTY

Department of Mathematics, University of Burdwan, Burdwan (W. B.)

(*Received* 16 *November* 1987; *after revision* 10 *April* 1989)

A uniform theory for a class of optimal control problems governed by linear plants with generalized constraints on the control variable is developed. The results obtained by the method of functional analysis provide new insight. Some necessary and sufficient conditions are derived to obtain explicit expression for the optimal control function.

Krasovskii[9] applied methods of functional analysis to solve a class of control problems. Kranc and Sarachik[8] and Kreindler[10] solved a class of optimization problems with various types of constraints. Hermes and Lasalle[6] also considered time optimal control problem from functional analysis point of view for amplitude constraints. Chaudhuri and Mukherjee[2] also developed a uniform theory of time optimal control problems with generalized constraints. They[3] also discussed the global controllability. Porter[13] demonstrated as to how the function space approach could be utilised to obtain the optimum control for a wide class of minimum norm controls. Minamide and Nakamura[11] generalised the minimum cost problem of Porter. Burns[1] also considered the minimum effort problem and minimum cost problem in the Banach space setting.

The main purpose of this paper is to generalize the idea of Porter in locally convex linear topological space, so that the more generalized constraints on the control variable can be treated. The explicit expression under the generalized constraints on the control variable is obtained.

Now, let L be a locally convex linear topological space and L^* be the conjugate space of all continuous linear functionals defined on L.

Let M^π be the polar set of $M \subseteq L$ in L^* (Yosida[14], p. 136). Then the boundary of M^π will be denoted by δM^π and is defined as follows :

$$\delta M^\pi = \{ f \in L^* : \sup_{l \in M} \{ | < l, f > | \} = 1 \}.$$

Again let E_π be the polar set of $E \subseteq L^*$ in $L \subset L^{**}$ (Yosida[14], p. 136). Then the boundary of E_π will be denoted by δE_π and is defined as follows :

$$\delta E_\pi = \{l \in L; \sup_{f \in E} \{ | < l, f > | \} = 1\}.$$

Also $E_\pi = E^{\pi}_{\pi\pi}$ for a non-void set $E \subseteq L^*$.

Here, we shall define the optimal control problem as follows : Let L and L' be locally convex linear topological spaces and T, a continuous linear transformation from L into L'. For each ξ in the range of T, we find an element $l \in L$ that satisfies $\xi = Tl$

while minimizing

$$J(l) = \sup_{f \in E} \{ | < l, f > | \}$$

where E is any subset of L^*.

To solve the above problem we shall consider E_π to be the polar set in L of any arbitrary bounded subset E of L^*. Furthermore we shall assume that

(i) E_π is bounded,

(ii) for every neighbourhood, say, U of zero in L, $T(U)$ contains some set that is of the second category in L' and that satisfies the condition of Baire.

Definition 1 (Reachable set)—The set of all points $\xi \in L'$ such that $\xi = Tl$ for some $l \in E_\pi$ will be called the reachable set with respect to the linear transformation T, and will be denoted by $C = T(E_\pi)$. Also, the boundary of C will be denoted by δc.

We characterize the reachable set of C as follows :

Theorem 1—The reachable set C is bounded, convex body and circled with respect to the origin in L'.

PROOF : Since E_π is convex and T is linear and since linear operators preserve convexity, then the reachable set C is convex. Also, since T is continuous linear transformation from L into L' and so bounded sets map into bounded sets, then by the hypothesis (i) the reachable set C is bounded. Again, if λ is any scalar with $| \lambda | \leqslant |$, then for $\xi \in C$ we shall show that $\lambda \xi \in C$. Now $\xi = Tl$ for some $l \in E_\pi$. Then $\lambda\xi = \lambda Tl = T(\lambda l) \in C$ for some $\lambda l \in E_\pi$ because $\sup_{f \in ECL^*} \{ | < \lambda l, f > | \} \leqslant | \lambda | \sup_{f \in ECL^*} \{ | < l, f > | \} \leqslant 1$. (Yosida[14], p. 136).

Thus C is circled. Now we see that the polar set E_π of any arbitrary bounded subset E of L^* is a neighbourhood of zero in the original topology of L and for that neighbourhood of zero E_π in L, $T(E_\pi)$ contains some set that is of the second category in L' and that satisfies the condition of Baire [by the hypothesis (ii)]. Therefore, from the

theorem on openness[7] (p 93), T is an open mapping onto L'. It follows that the open set is mapped into open set and hence C contains zero. Thus the reachable set C is a convex set with non-void interior and thus C is a convex body.

Theorem 2—If $\xi \in \delta C$, then the pre-image of ξ, say, l_ξ belongs to the boundary of E_π which must satisfy

$$\sup_{f \in ECL^*} \{ | < l_\xi, f > | \} = 1.$$

Proof : Let $\xi \in \delta C$ and let each pre-image of ξ belong to the interior of E_π for which $\sup_{f \in ECL^*} \{ | < l, f > | \} < 1$ (Yosida[14], p. 136). But, by hypothesis (ii) we can say that $C = T(E_\pi)$ contains some set that is of the second category in L' and that satisfies the condition of Baire for the neighbourhood E_π of zero in L. Therefore from theorem on openness[7] (p. 93) that T is an open mapping from L onto L' and so T maps open sets into open sets. Thus, it follows that the pre-image of ξ can not belong to the interior of E_π. Therefore, the pre-image of ξ, say, l_ξ belongs to the boundary of E_π which must satisfy $\sup_{f \in ECL^*} \{ | < l_\xi, f > | \} = 1$.

Now, if $l \in L$ maps into $\xi \in \delta C$ then

$$\sup_{f \in ECL^*} \{ | < l, f > | \} \geqslant 1.$$

Now the set of all pre-images of the vector $\xi \in L'$ will be denoted by the set inverse notation $T^{-1}(\xi)$. We shall show that existence of a minimum element of $T^{-1}(\xi)$ in the next theorem.

Theorem 3—Let $\xi \in \delta c$. Then $T^{-1}(\xi)$ has a minimum element if and only if $\xi \in c$.

Proof : Let $\xi \in c \cap \delta c$. Then $T^{-1}(\xi)$ contains an element of E_π. Since $\xi \in \delta c$, it follows from Theorem 2 that this element must satisfy $\sup_{f \in ECL^*} \{ | < l, f > | \} = 1$. Conversely, we suppose that l_ξ is a minimum element of $T^{-1}(\xi)$. Because T is homogeneous, it is clear that αl_ξ is a minimum element of $T^{-1}(\alpha\xi)$ for any scalar $\alpha > 0$. Now, if $\alpha < 1$ then $\alpha \xi \in C$ and so $\alpha\xi$ has a pre-image which satisfy

$$\sup_{f \in ECL^*} \{ | < \alpha l_\xi, f > | \} \leqslant 1 \text{ (Yosida}^{14}\text{, p. 136).}$$

Since α is arbitrary, it follows that $\sup \{ | < l_\xi, f > | \} \leqslant 1$ (Yosida[14], p. 136). Thus, $l_\xi \in \delta E_\pi$ and $\xi = T l_\xi \in C$.

Corollary 3.1—$T^{-1}(\xi)$ has a minimum element for each $\xi \in L'$ if and only if C is closed.

PROOF : If $T^{-1}(\xi)$ contains a minimum element for each $\xi \in L'$, then from Theorem 3 any $\xi \in \delta C$ implies $\xi \in C$. Therefore, $\xi \in C \cap \delta C$ implies that C is closed. Conversely we suppose that C is closed and p is the Minkowski functional of C. If $\xi \neq 0$ belongs to L' then $\xi \in p(\xi)\, C$ i.e., $\xi = p(\xi)\, x$ where $x \in C$ and so $x = p(\xi)^{-1} \xi \in C$. Also, $x \in \delta C$ because C is closed. Therefore $p(\xi)^{-1} \xi \in C \cap \delta C$. So, by Theorem 3, $p(\xi)^{-1} \xi$ has a minimum pre-image which implies that ξ has a minimum pre-image.

Corollary 3.2—If L is semi-reflexive, then the optimal control problem which has been described before can be solved for every continuous linear transformation T from L into L' and for prescribed hypothesis (i) and (ii).

PROOF : From the criterion for semi-reflexiveness it is known that each bounded weakly closed set is weakly compact[7] (p. 190). So, we assume that L is semi-reflexive. Then the polar set E_π of any arbitrary bounded subset E of L^* in L is weakly compact. Because E_π is closed and by the hypothesis (i) E_π is bounded. Again since T remains continuous when both L and L' are equipped with their weak topologies, then the continuous image of weakly compact set is weakly compact. Consequently the reachable set C is weakly compact and it is weakly closed and therefore strongly closed. Then the set C is closed. Therefore we can prove that the optimal control problem is solvable from Corollary 3.1.

Corollary 3.3—Let S be a continuous linear transformation from a locally convex linear topological space Y into another locally convex linear topological space X. Then the optimal control problem has a solution provided $S(Y)$ is closed when X is Hausdorff and the adjoint S^* of S is onto between X^* and Y^* i.e. for every $\xi \in L' = Y^*$ there is a pre-image $l_\xi \in L = X^*$ of ξ under $T = S^*$ with $\sup_{f \in ECL^*} \{ | < l_\xi, f > | \} = 1$.

PROOF : Since S is a continuous linear transformation of Y into X, then the adjoint S^* of S is also a continuous linear transformation from X^* into Y^*. Again since $S(Y)$ is closed when X is Hausdorff, it follows that S^* is an open mapping from X^* into Y^* (Kelley *et al.*[7], p. 204, Theorem 21.6). Also from hypothesis S^* is onto mapping. Furthermore, S^* is continuous when both X^* and Y^* are equipped with their weak* topology. Now it is known that the polar set M^π of $M \subset X$ where M is taken to be a convex balanced neighbourhood of zero in X, is weak* compact [application of Tychonov's theorem, (Yosida[14], p. 137)]. Therefore, we can conclude from the continuity of S^* that the reachable set $S^*(M^\pi)$ is closed and hence from Corollary 3.1 the optimal control problem is solvable for each $\xi \in L' = Y^*$ which has a minimum pre-image $l_\xi \in L = X^*$ with $\sup_{f \in ECL^*} \{ | < l_\xi, f > | \} = 1$ under the transformation $T = S^*$.

Now, applying application of Tychonov's theorem (Yosida[14], p. 137) and invoking Krein-Milman theorem on locally convex linear topological space (Yosida[14],

p. 362) we can infer that the polar set E_π^π in L^* of $E_\pi \subset L$ has at least one extreme point. Therefore, for every $0 \neq l \in L$ there exist at least one $f \in E_\pi^\pi \subset L^*$ such that $\sup_{l \in E_\pi C L} \{ | < l, f > | \} = 1$ and $f(l) = | < f, l > | = \sup_{f \in E_\pi{}^\pi C L^*} \{ | < f, l > | \}$.

Thus, such a vector $f \in E_\pi^\pi \subset L^*$ is called an extremal of $l \in L$.

Again, for every $0 \neq f \in L^*$ if there exists at least one $l \in E_\pi \subset L$ such that $f(l) = | < l, f > | = \sup_{l \in E_\pi} \{ | < l, f > | \}$ and $\sup_{f \in E C L^*} \{ | < f, l > | \} = 1$. Then such a vector $l \in E_\pi$ is called an extremal of $f \in L^*$.

Now we are to obtain the form of the optical control to the given optimal control problem.

Theorem 4—Let $\phi \in L'^*$ and let L^*_{w*} be the locally convex linear topological space with respect to the weak* topology of L^*. Corresponding to an element $T^* \phi \in L^*_{w*}$ if there exists at least one non-zero extremal l_ϕ then $T(l_\phi) \in C \cap \delta C$ holds. Conversely, each $\xi \in C \cap \delta C$ can be written as $T(l_\phi)$ where l_ϕ (i.e. it exists) is a non-zero extremal of $T^* \phi$ for some $\phi \in L'^*$.

Proof : We suppose that $\phi \in L'^*$ and suppose that $T^* \phi \in L^*_{w*}$ has a non-zero extremal $l_\phi \in E_\pi \subset L$.

We put $\xi_\phi = Tl_\phi$. Clearly, $\xi_\phi \in C$. Now, we are to show that $\xi_\phi \in \delta C$. Let any $\eta \in C$. Then $\eta = Tl$ for some $l \in E_\pi$ and consequently

$$
\begin{aligned}
& | < \eta, \phi > | = | < Tl, \phi > | = | < l, T^* \phi > | \\
& \leqslant \sup_{l \in E_\pi} \{ | < l, T^* \phi > | \} = | < l_\phi, T^* \phi > | = | < Tl_\phi, \phi > | \\
& = | < \xi_\phi, \phi > | \quad [\because \xi_\phi = Tl_\phi].
\end{aligned}
$$

It follows that the functional ϕ assumes its maximum value on C at the vector ξ_ϕ.

Again, since ϕ is continuous linear functional, open sets map into open sets. Thus, we show that ξ_ϕ can not belong to the interior of C. Therefore, $Tl_\phi \in C \cap \delta C$ for some $\phi \in L'^*$ and for $T^* \phi \in L^*_{w*}$.

Conversely, let $\xi \in C \cap \delta C$. Then from Theorem 3 we see that the minimum pre-images of ξ exist and we shall show that each pre-image of ξ is an extremal l_ϕ of $T^* \phi$ for some $\phi \in L'^*$. Now since $\xi \in \delta C$ and C is a convex body, we may consider that $\phi \in L'^*$ such that

$$\sup_{l \in E_\pi} \{ | < l, T^* \phi > | \}$$

$$\leqslant | < l, T^* \phi > | \text{ [for some } l \in \delta E_\pi]$$

$$= | < Tl, \phi > |$$

$$= | < \xi, \phi > | \; [Tl = \xi].$$

Again since $\xi \in C$, $\xi = TL$ for some $l \in E_\pi$ and so

$$\sup_{l \in E_\pi} \{ | < l, T^* \phi > | \} \geqslant | < l, T^* \phi > | \text{ for any } l \in E_\pi$$

$$= | < Tl, \phi > | = | < \xi, \phi > | .$$

These two inequalities show that

$$| < l, T^* \phi > | = | < \xi, \phi > | = \sup_{l \in E_\pi} \{ | < l, T^* \phi > | \}.$$

Again, since $l \in E_\pi$ and obviously $\sup\limits_{T^*\phi \in ECL^*} \{ | < l, T^* \phi > | \} = 1$, l is an extremal of $T^* \phi$ and so $\xi = Tl = Tl_\phi$ for some $\phi \in L'^*$ and for $T^* \phi \in L^*_{w*}$.

Corollary 4.1—If $\xi \in C \cap \delta C$, then $\xi = T(l_\phi)$ for some $\phi \in L'^*$ and for some $T^* \phi \in L^*_{w*}$ if and only if ϕ defines a supporting hyperplane to C at ξ.

PROOF : If $\xi \in C \cap \delta C$ and $\xi = T(l_\phi)$ for some $\phi \in L'^*$, and $T^* \phi \in L^*_{w*}$, then for each $l \in E_\pi$

$$| < Tl, \phi > | = | < l, T^* \phi > | \}$$

$$\leqslant \sup_{l \in E_\pi} \{ | < l, T^* \phi > | \}$$

$$= | < l_\phi, T^* \phi > |$$

$$= | < Tl_\phi, \phi > |$$

$$= | < \xi, \phi > | .$$

Consequently, we have the supporting hyperplane

$$\{\eta : | < \eta, \phi > | = | < Tl, \phi > | = | < \xi, \phi > | \} \text{ to } C \text{ at } \xi.$$

Again, if $\phi \in L'^*$ defines a supporting hyperplane at ξ, then proof is obvious from the converse part of Theorem 4.

Corollary 4.2—If $\xi \in C \cap \delta C$ and ϕ defines a supporting hyperplane to C at ξ, then $T^* \phi$ attains its supremum at $l_\phi \in E_\pi$. Conversely, if $T^* \phi$ attains its supremum at $l_\phi \in E_\pi$ for some $\phi \in L'^*$, then $T^{-1}(\xi_\phi)$ will contain a minimum element for which $\xi_\phi = Tl_\phi$.

PROOF : We suppose that ϕ supports C at ξ. Then from Corollary 4.1 and again from Theorem 4, $T^* \phi$ has at least one non-zero extremal l_ϕ for $\phi \in L'^*$. This shows that $T^* \phi$ attains its supremum at $l_\phi \in E_\pi$. Conversely, if $T^* \phi$ attains its supremum at $l_\phi \in E_\pi$ for some $\phi \in L'^*$, then by Theorem 4, $T(l_\phi) \in C \cap \delta C$. Hence by Theorem 3, $T^{-1}(\xi_\phi)$ has a minimum element l_ϕ for which $\xi_\phi = Tl_\phi$.

Thus, we can conclude that if L is semi-reflexive or the conditions of Corollary 3.3, then the Corollary 3.1 holds and combining Corollary 3.1, Corollary 4.2, and Theorem 4, we see that $T^{-1}(\xi)$ has minimum elements for every $\xi \in L'$ and each minimum element must have the form $l_\xi = p(\xi)\, l_\phi$ for some outward normal ϕ to C at $p(\xi)^{-1}\, \xi$ and some extremal l_ϕ of $T^* \phi \in L^*_{w*}$.

ACKNOWLEDGEMENT

The author is most grateful to Dr R. N. Mukherjee of the Department of Mathematics, University of Burdwan, Burdwan (W. B.) for his generous encouragement and suggestions at various steps while preparing this paper.

REFERENCES

1. J. A. Burns, *Indian J. pure appl. Math.*, **12** (1975), 151–62.
2. A. K. Chaudhuri and R. N. Mukherjee, *Indian J. pure appl. Math.*, **12** (1981), 151–62.
3. A. K. Chaudhuri and R. N. Mukherjee, *Indian J. pure appl. Math.*, **13** (1982), 163–71.
4. A. K. Chaudhuri and R. N. Mukherjee, *Problems of Control and Information Theory* **12** (3), (1983), 167–78.
5. N. Dunford and J. T. Schwartz, *Linear Operators,* I, Inter Science, New York, 1958.
6. H. Hermes and J. P. Lasalle, *Functional Analysis and Time Optimal Cortrol.* Academic Press, New York, 1969.
7. J. L. Kelley and Namioka Issac and co-authors, *Linear Topological Spaces* D. Van Nostand Co., Inc.
8. G. M. Kranc and P. E. Sarachik, *Trans. Am. Soc. Engrs. Basic Engineering*, **85**, 1963.
9. N. N. Krasovskii, *Auto. & Remote Control*, **18** (1957), 1005–16; *J. Appl. Math. Mech.*, **23** (1959), 899-919.
10. E. Kreindler, *J. Franklin Inst.*, **175** (1963), No. 4.
11. N. Minamide and K. A. Nakamura, *J. Math. Anal. Applic.* **36** (1971), 73-85.
12. L. S. Pontryagin, V. G. Bletyanskii, R. V. Gamkrelidze and E. F. Mischenko. *The Mathematical Theory of Optimal Processes.* Wiley & Sons, New York, 1962.
13. W. A. Porter, '*Modern Foundations of Systems Engineering*'. The Macmillan Co., New York, 1966.
14. Kôsaku Yosida, *Functional Analysis*, Springer-Verlag, Berlin, 1965.
15. I. Singer, *J. Math. Anal. Appl.* 69 (1979), 571–584.

Indian J. pure appl. Math., **20** (8) : 768-772, August 1989

BOUNDS FOR THE ZEROS OF POLYNOMIALS

M. Bidkham and K. K. Dewan

Department of Mathematics, Faculty of Natural Sciences and Technology
Jamia Millia Islamia, New Delhi 110025

(*Received* 24 *August* 1987; *after revision* 3 *October* 1988)

In this paper a ring shaped region containing all the zeros of the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ has been obtained. Our results improve upon a result proved by Deutsch [*Am. Math. Monthly* **88** (1981)], No. 3, and some well known classical results due to Cauchy.

1. Introduction and Statement of Results

Consider the polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \qquad \dots(1.1)$$

where $a_0, a_1, \dots, a_{n-1}$ are complex numbers.

For every zero z of $f(z)$ we have

$$|z| \leqslant \max\{|a_0|, 1 + |a_1|, \dots, 1 + |a_{n-1}|\} \qquad \dots(1.2)$$

$$|z| \leqslant \max\{1, |a_0| + |a_1| + \dots + |a_{n-1}|\} \qquad \dots(1.3)$$

$$|z| \leqslant r \qquad \dots(1.4)$$

where r is the unique positive zero of

$$g(z) = z^n - |a_{n-1}| z^{n-1} - \dots - |a_1| z - |a_0|. \qquad \dots(1.5)$$

The above classical results are basically due to Cauchy (see Marden[3], pp. 96-97, Parodi[4], p. 126) Deutsch[1] obtained new upper bounds for the absolute values of the zeros of $f(z)$, which generalize those given by (1.2), (1.3) and (1.4) by proving the following :

Theorem A—Every zero of the complex polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

satisfies

$$|z| \leqslant \max\{r_k, 1 + |a_{k+1}|, 1 + |a_{k+2}|, \dots, 1 + |a_{n-1}|\}$$

where $k \in \{0, 1, \dots, n-1\}$ and r_k is the unique positive zero of

$$g_k(z) = z^{k+1} - |a_k| z^k - |a_{k-1}| z^{k-1} - \dots - |a_1| z - |a_0| .$$

Theorem B—Every zero of the complex polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

satisfies

$$|z| \leqslant \max\{1, |a_0| + |a_1| + \dots + |a_k|, 1 + |a_{k+1}|, \dots, 1 + |a_{n-1}|\}$$

where

$$k \in \{0, 1, \dots, n-1\}.$$

In the present paper we shall improve upon Theorems A and B by obtaining a ring shaped region containing all the zeros of the polynomial. In fact we prove

Theorem 1—Let $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial complex coefficients then $f(z)$ has all its zeros in the ring-shaped region given by

$$R_2 \leqslant |z| \leqslant R_1.$$

Here

$$R_1 = \max\{r_k, 1 + |a_{k+1}|, 1 + |a_{k+2}| \dots, 1 + |a_{n-1}|\} \quad \dots (1.6)$$

where $k \in \{0, 1, \dots, n-1\}$ and r_k is the unique positive zero of

$$g_k(z) = z^{k+1} - |a_k| z^k - |a_{k-1}| z^{k-1} - \dots - |a_1| z - |a_0|$$

and

$$R_2 = \frac{1}{2M_1^2}[-R_1^2 |b| (M_1 - R_1 |a_0|) + \{R_1^4 |b|^2 (M_1 - R_1 |a_0|)^2 + 4|a_0| R_1^3 M_1^3\}^{1/2}] \quad \dots (1.7)$$

where

$$M_1 = R_1^{n+1}\left(2 + \sum_{k=1}^{n-1} |a_k| + \frac{1}{R_1}\sum_{k=0}^{n-1} |a_k|\right)$$

$$b = |a_1| R_1 - |a_0| . \quad \dots (1.8)$$

Theorem 2—Let $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial with complex coefficients, then $f(z)$ has all its zeros in the ring shaped region given by

$$R_4 \leqslant |z| \leqslant R_3.$$

Here

$$R_3 = \max\{1, |a_0| + |a_1| + \ldots + |a_k|,$$
$$1 + |a_{k+1}|, \ldots, 1 + |a_{n-1}|\} \qquad \ldots(1.9)$$

where $k \in \{0, 1, \ldots, n-1\}$ and

$$R_4 = \frac{1}{2M_2^2}\,[-R_3^2\,|b|\,(M_2 - R_3|a_0|) + \{R_3^4\,|b|^2\,(M_2 - R_3$$
$$|a_0|)^2 + 4\,|a_0|\,R_3^3\,M_2^3\}^{1/2}] \qquad \ldots(1.10)$$

and

$$M_2 = R_3^{n+1}\left(2 + \sum_{k=1}^{n-1}|a_k| + \frac{1}{R_3}\sum_{k=0}^{n-1}|a_k|\right) \qquad \ldots(1.11)$$

$$b = |a_1|\,R_3 - |a_0|.$$

Remark 1: It is easy to see that for $k = 0$ and $k = n - 1$, Theorem A fails to give an improvement of inequalities (1.2) and (1.4) while our Theorem 1 improves upon the inequalities (1.2) and (1.4) respectively, in this case also.

Remark 2: Again for $k = 0$ and $k = n - 1$, Theorem B yields the inequalities (1.2) and (1.3) respectively while Theorem 2 obviously improves the inequalities (1.2) and (1.3) respectively.

Example—Consider the polynomial

$$f(z) = z^3 + 0.3\,z^2 + 0.7\,z + 0.7.$$

Then by inequalities (1.2) and (1.3), $f(z)$ has all its zeros in $|z| \leqslant 1.7$, by Theorem B, with $k = 1$, $f(z)$ has all its zeros in $|z| \leqslant 1.4$ whereas by Theorem 2, with $k = 1$, $f(z)$ has all its zeros in

$$0.67 \leqslant |z| \leqslant 1.4$$

2. Lemmas

Lemma 1—If $f(z)$ is analytic in $|z| \leqslant 1$, $f(0) = a$, where $|a| < 1$, $f'(0) = b$, $|f(z)| \leqslant 1$ on $|z| = 1$, then for $|z| \leqslant 1$,

$$|f(z)| \leqslant \frac{(1-|a|)\,|z|^2 + |b|\,|z| + |a|\,(1-|a|)}{|a|\,(1-|a|)\,|z|^2 + |b|\,|z| + (1-|a|)}. \qquad \ldots(2.1)$$

The example $f(z) = \left(a + \frac{b}{1+a}z - z^2\right)\Big/\left(1 - \frac{b}{1+a}z - az^2\right)$ shows that the estimate is sharp.

The above lemma is due to Govil, *et al.*[2]. One gets easily from Lemma 1, the following

Lemma 2—If $f(z)$ is analytic in $|z| \leqslant R$, $f(0) = 0$, $f'(0) = b$. and $|f(z)| \leqslant M$ for $|z| = R$, and then for $|z| \leqslant R$

$$|f(z)| \leqslant \frac{M|z|}{R^2} \frac{M|z| + R^2|b|}{M + |z|\,|b|}. \qquad \ldots(2.2)$$

3. Proofs of Theorems

Proof of Theorem 1. In view of Theorem A it is sufficient to prove that $f(z) \neq 0$ if

$$|z| < R_2$$

where R_2 is defined in (1.7).

For arbitrary $\gamma \in \mathbb{R}$ we consider

$$F(z) = (R_1 - z)\, f(e^{i\gamma} z)$$

$$= R_1 a_0 + \sum_{k=1}^{n} (R_1 a_k e^{i\gamma} - a_{k-1})\, e^{(k-1)i\gamma} z^k$$

$$- e^{ni\gamma} z^{n+1},\; a_n = 1$$

$$= R_1 a_0 + P(z), \text{ say.} \qquad \ldots(3.1)$$

Clearly

$$|P(z)| \leqslant |z|^{n+1} + \sum_{k=1}^{n} |R_1 e^{i\gamma} a_k - a_{k-1}|\; |z|^k,\; a_n = 1$$

and hence

$$M(R_1) := \max_{|z|=R_1} |P(z)|$$

$$\leqslant R_1^{n+1} + R_1^n \sum_{k=1}^{n} |R_1 e^{i\gamma} a_k - a_{k-1}|,\; a_n = 1$$

$$\leqslant 2R_1^{n+1} + R_1^{n+1} \sum_{k=1}^{n-1} |a_k| + R_1^n \sum_{k=0}^{n-1} |a_k|$$

$$= R_1^{n+1} \left(2 + \sum_{k=1}^{n-1} |a_k| + \frac{1}{R_1} \sum_{k=0}^{n-1} |a_k| \right)$$

$$= M_1, \text{ say.} \qquad \ldots(3.2)$$

Now we note that $P(0)=0$, $|P'(0)| = |R_1 a_1 e^{i\gamma} - a_0| = |b|$ if γ is appropriately chosen. Choosing γ subject to this requirement and then applying Lemma 2 we obtain

$$|P(z)| \leqslant \frac{M_1 |z|}{R_1^2} \frac{M_1 |z| + R_1^2 |b|}{M_1 + |b| |z|} \qquad \text{...(3.3)}$$

for $|z| \leqslant R_1$.

Combining (3.1) and (3.3), we get, for $|z| \leqslant R_1$

$$|F(z)| \geqslant R_1 |a_0| - \frac{M_1 |z|}{R_1^2} \frac{M_1 |z| + R_1^2 |b|}{M_1 + |b| |z|}$$

$$= \frac{-1}{R_1^2 (M_1 + |b| |z|)} \{ |z|^2 M_1^2 + R_1^2 |b| |z|$$

$$\times (M_1 - R_1 |a_0|) - |a_0| R_1^3 M_1\}$$

$$> 0$$

if

$$|z| < \frac{-R_1^2 |b| (M_1 - R_1 |a_0|) + \{R_1^4 |b|^2 (M_1 - R_1 |a_0|)^2 + 4 |a_0| R_1^3 M_1^3\}^{1/2}}{2M_1^2}$$

$$= R_2.$$

Since the zeros of $f(ze^{i\gamma})$ lie exactly in the same disc (centred at the origin) as the zeros of $f(z)$ the polynomial $f(z)$ has no zeros in

$$|z| < R_2.$$

This complets the proof of Theorem 1.

We omit the proof of Theorem 2 as it is analogous to that of Theorem 1 except that we have to use Theorem B instead of Theorem A.

References

1. E. Deutsch, *Am. Math. Monthly*, 88 (1981), 205–206.
2. N. K. Govil, Q. I. Rahman and G. Schmeisser, *Ill. J. Math.* 23 (1979), 319–29.
3. M. Marden, *Am. Math. Soc.* 1949,
4. M. Parodi, *La Localisation des Valeurs Caracteristiques de Matrices et ses Applications*, Gauthier-Villars. Paris, 1959.

Indian J. pure appl. Math., **20** (8) : 773–781, August 1989

SELECTION PROCEDURES FOR HAZARD RATES BASED ON TWO-SAMPLE STATISTICS

AMAR NATH GILL AND GOBIND P. MEHTA

Department of Statistics, Panjab University, Chandigarh 160014

(*Received* 24 *February* 1988; *after revision* 20 *January* 1989)

Let $\pi_1, \pi_2, \cdots, \pi_k$ be k independent populations and F_i be the absolutely continuous cdf (cumulative distribution function) with continuous pdf (probability density function) as f_i of the life time of individuals in the ith population, $i = 1, 2, \ldots k$. We assume that the hazard rate $r_i(x) = f_i(x)/\bar{F}_i(x)$, $x \geqslant 0$ exists for each $i = 1, 2, \ldots, k$. Here $\bar{F}_i(x) = 1 - F_i(x)$ is the survival function. In this paper we consider the problem of selecting a subset of k populations containing the one associated with uniformly smaller hazard rate. The proposed selection procedures are based on two sample statistics which are functions of ordered ranks of the two samples uuder consideration. The procedure in general has may desirable properties.

1. INTRODUCTION

Suppose we have k independent populations of life lengths of their individuals which are being ranked in terms of hazard (failure) rates. We propose in this paper subset selection procedures to select a subset containing the population associated with uniformly smaller hazard rate based on two sample statistics given by Cheng[1]. Earlier, in an abstract, Patel[2] suggested a procedure for the problem of selecting a subset containing the population with the largest failure rate average out of several IFRA populations based on the number of failures observed by some fixed common time T. To the best of our knowledge no such paper appeared in the literature thereafter.

In section 2 of this paper we have formulated the problem. Proposed selection procedures are given in section 3. Section 4 deals with the probability of a correct selection, expected subset size and infimum of a probability of correct selection. In section 5 some desirable properties of the proposed procedure are discussed.

2. FORMULATION OF THE PROBLEM

Let $\pi_1, \pi_2, \ldots, \pi_k$ be k independent populations and F_i be the absolutely continuous cdf (cumulative distribution function) with continuous pdf (probability density function) as f_i of the life time of individuals in the ith population, $i = 1, \ldots, k$. Let the hazard rate $r_i(x) = f_i(x)/\bar{F}_i(x)$, $x \geqslant 0$ exist for each i, $i = 1, 2, \ldots, k$. Here $\bar{F}_i(x) = 1 - F_i(x)$ is the survival function of ith population, $i = 1, 2, \ldots, k$. Let

$\Omega = \{\mathbf{r} : \mathbf{r} = (r_1, r_2, \ldots, r_k)^t, r_i(x) < \infty, x \geqslant 0$ for all $i = 1, 2, \ldots, k\}$ be the space of hazard rates of the life times of individuals of k populations. For any two population π_i and π_j, π_i is considered to be better than π_j if $r_i(x) \leqslant r_j(x) \;\forall\; x \geqslant 0$. We assume that there is a best population, that is, for each $\mathbf{r} \in \Omega$, there is an index i such that $r_i(x) \leqslant r_j(x)$, $x \geqslant 0$ and $\forall\; j, j \neq i$. If more than one populations are tied for the best then arbitrarily one of them is labelled as the best.

For any $\mathbf{r} = (r_1, r_2, \ldots, r_k)^t$, we shall denote by $r_{[1]}$ the unique component of $\mathbf{r}$ corresponding to the best population. The goal is to select a subset of k populations containing the best population, the one with uniformly smaller hazard rate $r_{[1]}$. Any such selection will be called as CS (correct selection). Then the problem is to find a rule R such that for a pre-assigned probability P^* $(\frac{1}{k} < P^* < 1)$, this satisfies the probability requirement :

$$P_{\mathbf{r}}[CS \mid R] \geqslant P^* \;\forall\; \mathbf{r} \in \Omega. \qquad \ldots(2.1)$$

Let A be the action space of the subset selection problem which is the set of all nonempty subsets of $\{1, 2, \ldots, k\}$, where taking action $a \in A$ means the selection of those populations whose indices are in a. For any $a \in A$, let

$$CS(\mathbf{r}, a) = \begin{cases} 1 \text{ if } r_{[1]} \in \{r_i \,;\, i \in a\} \\ 0 \text{ otherwise} \end{cases} \qquad \ldots(2.2)$$

and $|a|$ = number of elements in a.

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$ be a random sample of size n_i from the ith population, $i = 1, 2, \ldots, k$. Let $\mathbf{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})^t$ be the vector of observations from the ith population and let $\mathbf{X} = (X_{11}, X_{12}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}, \ldots, X_{k1}, \ldots, X_{kn_k})^t$ be the vector of all the observations. For any subset selection procedure R, let $Z_R(\mathbf{X}, a)$ be the probability assigned to a by R having observed $\mathbf{X}$.

3. Proposed Selection Procedure

We shall now develop procedures for the above problem on the basis of the estimators of the parameters

$$\Delta(F_i, F_j) = \frac{\int_0^\infty \{\bar{F}_i(x)/[\bar{F}_i(x) + \bar{F}_j(x)]^{1/2}\}\, d\bar{F}_j(x)}{\int_0^\infty \{\bar{F}_j(x)/[\bar{F}_i(x) + \bar{F}_j(x)]^{1/2}\}\, d\bar{F}_i(x)}, \; i \neq j.$$

Such parameters have been proposed by Cheng[1].

For the ith and the jth population

$$
\begin{aligned}
& r_i(x) \leqslant r_j(x) \ \forall \ x \geqslant 0 \\
& \Leftrightarrow f_i(x)/\bar{F}_i(x) \leqslant f_j(x)/\bar{F}_j(x) \ \forall \ x \geqslant 0, \\
& \Leftrightarrow \bar{F}_j(x) f_i(x) \leqslant \bar{F}_i(x) f_j(x) \ \forall \ x \geqslant 0, \\
& \Leftrightarrow \{\bar{F}_j(x)/[\bar{F}_i(x) + \bar{F}_j(x)]^{1/2}\} f_i(x) \leqslant \{\bar{F}_i(x)/[\bar{F}_i(x) + \bar{F}_j(x)]^{1/2}\} \\
& \qquad f_j(x) \ \forall \ x \geqslant 0, \\
& \Rightarrow \int_0^\infty \{\bar{F}_i(x)/[\bar{F}_i(x) + \bar{F}_j(x)]^{1/2}\} f_j(x)\, dx \\
& \qquad\qquad \geqslant \int_0^\infty \{\bar{F}_j(x)/[\bar{F}_i(x) + \bar{F}_j(x)]^{1/2}\} f_i(x)\, dx \\
& \Leftrightarrow \Delta(F_i, F_j) \geqslant 1. \qquad \ldots(3.1)
\end{aligned}
$$

Equality in (3.1) holds if $F_i(x) = F_j(x)$ for all $x \geqslant 0$.

To obtain the estimator of $\Delta(F_i, F_j)$, we replace $\bar{F}_i$ and $\bar{F}_j$ by the respective emperical survival functions. Let $n_i + n_j = n$ and $S^i_{(1)} \leqslant S^i_{(2)} \leqslant \ldots \leqslant S^i_{(n_i)}$ and $R^j_{(1)} \leqslant \ldots \leqslant R^j_{(n_j)}$ be respectively the ordered ith and jth sample ranks in the combined sample. The estimator of $\Delta(F_i, F_j)$ is $T_{ij} = T_{jn}/T_{in}$, where

$$
\begin{aligned}
T_{jn} = \sum_{\alpha=1}^{n_j-1} \Big[\Big\{ n_i + \alpha - R^j_{(\alpha)} \Big\} \Big/ \Big\{ 2n_j^2\ n_i^2 + n_j^2\ n_i\, \alpha - n_j\ n_i^2\ \alpha \\
- n_j^2\ n_i\ R^j_{(\alpha)} \Big\}^{1/2} \Big] + \Big\{ n^{3/2} - n^{1/2}\ R^j_{(n_j)} \Big\} \Big/ \\
\Big\{ n_i^2\ n_j^2 + n_j^2\ n_i\, n^2 - n_j^2\ n_i\ n\ R^j_{(n_j)} \Big\}^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
T_{in} = \sum_{\beta=1}^{n_i-1} \Big[\Big\{ n_j + \beta - S^i_{(\beta)} \Big\} \Big/ \Big\{ 2n_i^2\ n_j^2 + n_j\ n_i^2\ \beta \\
- n_j^2\ n_i\, \beta - n_i^2\ n_j\ S^i_{(\beta)} \Big\}^{1/2} \Big] + \Big\{ n^{3/2} - n^{1/2}\ S^i_{(n_i)} \Big\} \Big/ \\
\Big\{ n_i^2\ n_j^2 + n_j\, n_i^2\ n^2 - n_j\, n_i^2\ n\ S^i_{(n_i)} \Big\}^{1/2}.
\end{aligned}
$$

The statistics T_{ij}, proposed by Cheng[1] for testing $r_i(x) = r_j(x) \ \forall \ x \geqslant 0$ against $r_i(x) \leqslant r_j(x) \ \forall \ x \geqslant 0$, depends only on the ordered ranks, which are uniquely defined, since F_i and F_j are both continuous. For tied observations average ranks can be used. Our selection procedure is based on T_{ij} and is defined as follows :

R_a : Select π_i in the subset iff

$$T_{ij} \geqslant C_j^{(i)}(\mathbf{n}, P^*) \ \forall \ j, j \neq i. \qquad \text{...(3.2)}$$

The constants $C_j^{(i)}(\mathbf{n}, P^*) \geqslant 0$ are chosen such that

$$P_0\,[T_{ij} \geqslant C_j^{(i)}(\mathbf{n}, P^*) \ \forall \ j, j \neq i] \geqslant P^*. \qquad \text{...(3.3)}$$

Here P_0 indicates that the probability is computed under $r_1 \equiv r_2 \equiv \ldots \equiv r_k$ and $\mathbf{n} = (n_1, n_2, \ldots, n_k)^t$.

The procedure (3.2) can be modified to select a subset of k populations better than the unknown control population. Let $r_0(x) = f_0(x)/\bar{F}_0(x)$ be the hazard rate of the control population π_0 and $r_i(x) = f_i(x)/\bar{F}_i(x)$ be the hazard rate of the population π_i, $i = 1, 2, \ldots, k$. Population π_i is considered to be better than π_0 if $r_i(x) \leqslant r_0(x)$ for all $x \geqslant 0$. Let n_j be the number of observations taken from population π_j, $j = 0, 1, 2, \ldots, k$ and let $\mathbf{n}^* = (n_0, n_1, \ldots, n_k)^t$. The proposed selection procedure based on the statistic T_{i0}, the estimator of parameter $\Delta(F_i, F_0)$, is

R_{a_1} : Select π_i in the subset if and only if

$$T_{i0} \geqslant C_0^{(i)}(\mathbf{n}^*, P^*). \qquad \text{...(3.4)}$$

The nonnegative constants $C_0^{(i)}(\mathbf{n}^*, P^*)$ are chosen such that

$$P_0\,[T_{i0} \geqslant C_0^{(i)}(\mathbf{n}^*, P^*), i = 1, 2, \ldots, k] \geqslant P^*.$$

Here P_0 indicates that the probability is computed under $r_0(x) = r_1(x) = \ldots = r_k(x)$ for ail $x \geqslant 0$.

The procedure (3.4) can approximately be used with the help of existing tables for the case of large and moderate equal sample sizes from all the $(k + 1)$ populations as follows :

Let $n_0 = n_1 = \ldots = n_k = n$. By Theorem 2.1 of Cheng[1], it follows that as $N \to \infty$ such that $n/N \to p$ (here $N = n(k + 1)$), the limiting distribution of

$$Z_{i0} = 2\sqrt{n}\,(T_{i0} - 1)/3$$

is normal with mean zero and variance 1 if $r_i(x) = r_0(x)$ for all $x \geqslant 0$. It is a known fact that the joint distribution of standardized correlated variables often tends asymptotically to multivariate normal distribution (c.f. Gupta *et al.*[3]). Thus the limiting distribution of the random vector $\mathbf{Z} = (Z_{10}, \ldots, Z_{k0})^t$ under $r_0(x) = r_1(x) = \ldots = r_k(x)$ for all $x \geqslant 0$ will be asymptotically multivariate normal of equally correlated standard variables when $n_0 = n_1 = \ldots = n_k = n$. Now the constants $C_0^{(i)}(\mathbf{n}^*, P^*)$ are determined such that

$$\begin{aligned} P^* &= P_0\,[Z_{i0} \geqslant C_0 \text{ for all } i = 1, 2, \ldots, k] \\ &= P_0\,[\min_i Z_{i0} \geqslant C_0] \\ &= P_0\,[\max_i Z_{i0} \leqslant -C_0]. \end{aligned}$$

Here

$$C_0 = 2\sqrt{n}\,[C_0^{(i)}(\mathbf{n}^*, P^*) - 1]/3.$$

Now we can use the Table I of Gupta *et al.*[3] to read the constant $-C_0$ and thereby get the values of constants $C_0^{(i)}(\mathbf{n}^*, P^*)$, $i = 1, 2, \ldots, k$.

4. Probability of Correct Selection and Expected Subset Size

Let us define, $\tilde{A} = \{a \in A \mid CS(\mathbf{r}, a) = 1\}$, the probability of a correct selection is then

$$\begin{aligned} P_{\mathbf{r}}[CS \mid R_a] &= P\;[r_{[1]} \text{ is in the selected subset} \mid R_a] \\ &= P_{\mathbf{r}}\,[\bigcup_{a \in \tilde{A}} (\mathbf{X} \text{ is observed and action } a \text{ is taken}) \mid R_a] \\ &= \int_0^\infty P_{\mathbf{r}}\,[\bigcup_{a \in \tilde{A}} \{\text{action } a \text{ is taken} \mid \mathbf{X} = \mathbf{x}, R_a\}]\, dF(\mathbf{x}) \end{aligned}$$

There may be different subsets in A which contain the best population. All these subsets form $\tilde{A}$. However, on the basis of available observations one and only one subset from A can be chosen, that is, only one action at a time is possible. Thus the above integral may be written as

$$E_{\mathbf{r}}\,[\sum_{a \in \tilde{A}} P_{\mathbf{r}}(\text{action } a \text{ is taken} \mid \mathbf{X}, R_a)]$$

(*equation continued on p. 778*)

$$= E_r[\sum_{a \in A} CS(\mathbf{r}, a)\, Z_{R_a}(\mathbf{X}, a)].$$

The expected subset size is

$$E_r[S \mid R_a] = E_r[\sum_{a \in A} |a|\, Z_{R_a}(\mathbf{X}, a)].$$

For fixed P^*, $0 \leqslant P^* \leqslant 1$, a procedure R is said to satisfy the P^* condition if

$$\inf_{r \in \Omega} P_r[CS \mid R] \geqslant P^*.$$

Let $P^* > \frac{1}{k}$ be specified. The following lemma due to Cheng[1] will be used to show that the procedure R_a satisfies the P^* condition.

Lemma 4.1—Assume that the statistic $S(x_{j1}, \ldots, x_{jn_j}, x_{i1} \ldots, x_{in_i})$ is symmetric in arguments $\{x_{j\alpha}, \alpha = 1, \ldots, n_j\}$ and $S(x_j^{(1)}, \ldots x_j^{(n_j)}; x_{i1}, \ldots, x_{in_i}) \leqslant S(\dot{x}_j^{(1)}, \ldots, \dot{x}_j^{(n_j)}; x_{i1}, \ldots, x_{in_i})$ for every $\dot{x}_j^{(\alpha)} \leqslant x_j^{(\alpha)}$, $\alpha = 1, 2, \ldots, n_j$ and $x_{i\beta}$'s, $\beta = 1, 2, \ldots, n_i$. Here $x_j^{(\alpha)}$ denotes the αth order statistic from the jth sample. If $\bar{F}_i(x) \geqslant \bar{F}_j(x)$ for all $x \geqslant 0$ and F_i and F_j are both continuous distributions then for the random samples $\{X_{i\beta}, \beta = 1, \ldots, n_i\}$ and $\{X_{j\alpha}, \alpha = 1, 2, \ldots, n_j\}$ and every constant C^*,

$$P[S(X_{j1}, \ldots, X_{jn_j}; X_{i1}, \ldots X_{in_i}) \geqslant C^* \mid X_{i\beta} \sim F_i; X_{j\alpha} \sim F_i]$$

$$\leqslant P[S(X_{j1}, \ldots, X_{jn_j}; X_{i1}, \ldots, X_{in_i}) \geqslant C^* \mid X_{i\beta} \sim F_i, X_{j\alpha} \sim F_j].$$

Theorem 4.2—For the procedure R_a, P^* condition is satisfied.

PROOF: Assume without loss of generality, that π_i is the best population i.e., $r_i(x) \leqslant r_j(x)$ for all j $(j \neq i)$ and $x \geqslant 0$. This implies $\bar{F}_i(x) \geqslant \bar{F}_j(x)$ for all j $(j \neq i)$ and $x \geqslant 0$. The statistic T_{ij} satisfies the conditions of the above lemma. Hence

$$P^* \leqslant P_0[T_{ij} \geqslant C_j^{(i)}(\mathbf{n}, P^*)\ \forall\ j, j \neq i]$$

$$\leqslant P_r[T_{ij} \geqslant C_j^{(i)}(\mathbf{n}, P^*) \text{ for all } j, j \neq i].$$

5. Properties of Procedure R_a

Gupta and Nagel[4] and Santner[5] have defined some desirable properties of a selection procedure, viz. unbiasedness, monotonicity, strong monotonicity while proposing selection procedures for parametrtc families. We see below that with necessary

modifications in the definitions, these properties also hold for our procedure. In what follows we define and prove the strong monotonicity of R_a, which implies its monotonicity and hence the unbiasedness. For any $\mathbf{r} \in \Omega$, let

$$p_{\mathbf{r}}^{\mathbf{n}}(i) = P_{\mathbf{r}}[R_a \text{ selects } \pi_i] \text{ for } i = 1, 2, .., k.$$

Definition 5.1—The procedure R_a is strongly monotone in π_i means

$$\begin{aligned} p_{\mathbf{r}}^{\mathbf{n}}(i) \text{ is } &\downarrow \text{ in } r_i(x) = f_i(x)/\bar{F}_i(x) \text{ when all other} \\ &\quad \text{components of } \mathbf{r} \text{ are fixed;} \\ \text{is } &\uparrow \text{ in } r_j(x)\ (j \neq i) \text{ when all other} \\ &\quad \text{components of } \mathbf{r} \text{ are fixed.} \end{aligned}$$

To show the strong monotonicity of R_a we prove the following lemmas :

Lemma 5.1—The family of distributions of T_{ij} is stochastically increasing.

PROOF : Let $r_i(x) < r_0(x) < r_j(x)\ \forall\ x \geqslant 0$ which implies $\Delta(F_i, F_j) > \Delta(F_0, F_j)$ and let $G(x; \Delta(F_i, F_j)) = P_{\Delta(F_i,\ F_j)}[T_{ij} \leqslant x]$ be the cdf of T_{ij}. We want to show that $G(x; \Delta(F_i, F_j)) \leqslant G(x; \Delta(F_0, F_j))\ \forall\ x > 0$. Now $r_i(x) < r_0(x) < r_j(x) \Rightarrow F_i(x) \leqslant F_0(x) \leqslant F_j(x)\ \forall\ x \geqslant 0$. The result follows from Theorem 4.3.3 of Randles and Wolfe[6] by taking $H \equiv F_i$, $G \equiv F_0$ and $F \equiv F_j$.

Lemma 5.2—Let $G(x)$ and $H(x)$ be two absolutely continuous distributions with $G(0) = H(0) = 0$ and let $r_G(x)$ and $r_H(x)$ be the corresponding hazard rates. If $r_H(x) \leqslant r_G(x)\ \forall\ x \geqslant 0$ and $S(x)$ is a nondecreasing function of x, then

$$E_H[S(X)] \geqslant E_G[S(X)].$$

PROOF : Since, $r_H(x) \leqslant r_G(x) \Rightarrow H(x) \leqslant G(x)\ \forall\ x \geqslant 0$, the lemma follows.

Lemma 5.3—Let $X_1, X_2, \ldots, X_k$ be k independent random variables where distribution of X_i is F_i with hazard rate $r_i(x)$, $x \geqslant 0$, $i = 1, 2, \ldots, k$. For any fixed i $(1 \leqslant i \leqslant k)$ if $S(x_1, x_2, \ldots, x_k)$ is a nondecreasing function of x_i when all x_j, $j \neq i$, are held fixed and $r_G(x)$ be the hazard rate corresponding to a continuous distribution $G(x)$ such that $r_i(x) \leqslant r_G(x)$, $x \geqslant 0$, then

$$E_{r_1, r_2, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_k}[S(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_k)]$$

$$\geqslant E_{r_1, r_2, \ldots, r_{i-1}, r_G, r_{i+1}, \ldots, r_k}[S(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_k)].$$

PROOF : $r_i(x) \leqslant r_G(x)\ \forall\ x \geqslant 0 \Rightarrow F_i(x) \leqslant G(x)\ \forall\ x \geqslant 0$.

Now

$$E_{r_1, \ldots, r_i, \ldots, r_k} [S(X_1, \ldots, X_i, \ldots, X_k)]$$

$$= E_{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k} [E_{r_i} \{S(x_1, x_2, \ldots, x_{i-1}, X_i, x_{i+1}, \ldots, x_k)\}]$$

and $S(x_1, x_2, \ldots, x_i, \ldots, x_k)$ is a nondecreasing function of x_i. Hence by Lemma 5.2 it follows that

$$E_{r_i} \{S(x_1, \ldots, x_{i-1}, X_{i+1}, \ldots\ldots, x_k)\}$$

$$\geqslant E_{r_G} \{S(x_1, x_2, \quad, x_{i-1}, X_i, x_{i+1}, \ldots, x_k)\}.$$

Since this is true for each fixed i, the lemma follows.

Theorem 5.1—The procedure R_a defined in (3.2) is strongly monotone in π_i, for any $i = 1, 2, \ldots, k$.

PROOF: The family of distribution of T_{ij} is stochastically increasing and for each fixed i, $\min_{j \neq i} T_{ij}$ is nondecreasing in $\mathbf{X}_i$, when other components of $\mathbf{X}$ are held fixed. For fixed i $(1 \leqslant i \leqslant k)$ define.

$$Q(T_{ij}) = \begin{cases} 1 \text{ if } T_{ij} \geqslant C_j^{(i)} (\mathrm{n}, P^*) \text{ for all } j \neq i \\ 0 \text{ otherwise.} \end{cases}$$

If $r_i(x) \leqslant r_i^*(x)$ and r_j $(j \neq i)$ are held fixed, it follows from Lemma 5.3, that

$$E_{r_1, \ldots, r_i, \ldots, r_k} [Q(T_{ij})] \geqslant E_{r_1, \ldots r_i^*, \ldots, r_k} [Q(T_{ij})]$$

$$\Rightarrow P_{\mathrm{r}} [R_a \text{ selects } \pi_i] \geqslant P_{\mathrm{r}^*} [R_a \text{ selects } \pi_i]$$

i.e.

$$p_{\mathrm{r}}^{\mathrm{n}}(i) \geqslant p_{\mathrm{r}^*}^{\mathrm{n}}(i) \qquad \ldots(5.1)$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_i, \ldots, r_k)^t$ and

$$\mathbf{r}^* = (r_1, r_2, \ldots, r_i^*, \ldots, r_k)^t.$$

(5.1) implies that the selection procedure R_a is strongly monotone.

Since a selection procedure which is strongly monotone is also monotone (see, for example Santner[5]), and hence unbiased, we have the following corollary.

Corollary 5.1—The selection procedure R_a is monotone and unbiased.

Acknowledgement

Thanks are due to the referee for suggesting improvements in the earlier version of the paper.

References

1. K. F. Cheng, *Biometrika* **72** (1985), 211–15.
2. J. K. Patel, *Ann. Math. Statist.* **38** (1967), 957.
3. S. S. Gupta, K. Nagel, and S. Panchapakesan, *Biometrika* **60** (1973), 403–13.
4. S. S. Gupta and K. Nagel, In *Statistical Decision Theory and Related Topics* (ed. : S. S. Gupta and J. Yackel). Academic Press, New York, 1971, pp. 79–102.
5. T. J. Santner, *Ann. Statist.* **3** (1975), 334–49.
6. R. H. Randles and D. A. Wolfe, *Introduction to the Theory of Nonparametric Statistics*. Wiley, New York, 1979.
7. D. M. Mahamunulu, *Ann. Math. Statist.* **38** (1967), 1079–91.

Indian J. pure appl. Math., **20** (8) : 782–785, August 1989

ON DIGRAPH RECONSTRUCTION

S. RAMACHANDRAN

Department of Mathematics, Vivekananda College, Agasteeswaram 629701
Kanyakumari

(*Received* 24 *February* 1988; *after revision* 16 *January* 1989)

We reconstruct a class of digraphs and improve a result on the reconstruction of graphs due to P. Z. Chinn.

1. INTRODUCTION

A digraph D consists of a finite set V of points and a set A of ordered pairs of distinct points. Any such pair (u, v) is called an arc from u to v. With each point v of a digraph D, we can associate a triple (r, s, k), called the degree triple of v, where $r = |\{u \mid (v, u) \in A \text{ and } (u, v) \notin A\}|$, $s = |\{u \mid (u, v) \in A \text{ and } (v, u) \notin A\}|$, and $k = |\{u \mid (u, v) \in A \text{ and } (v, u) \in A\}|$. We use the terminology as given in Harary[2].

Reconstruction Conjecture—Any graph with at least three points can be reconstructed up to isomorphism, from the collection of its point-deleted subgraphs.

Digraph Reconstruction Conjecture (DRC)—Any digraph with at least five points can be reconstructed, up to isomorphism, from the collection of its point-deleted subdigraphs.

If D is a digraph with points $v_1, v_2, \ldots v_n$ and d_i is the degree triple of v_i in D, then $(D - v_i, d_i)$ is called a degree triple associated point-deleted subdigraph of D.

New Digraph Reconstruction Conjecture (NDRC)[3]—Any digraph can be reconstructed, up to isomorphism, from the collection of its degree triple associated point-deleted subdigraphs.

A digraph is called reconstructible if it obeys DRC and N-reconstructible if it obeys NDRC. It is obvious that the truth of DRC implies the truth of NDRC, which in turn implies the truth of the reconstruction conjecture. Stockmeyer[5] has found six infinite families of counterexamples to the DRC, but the digraphs in all these counterexamples obey NDRC[4]. In fact, no counterexample has so far been unearthed for NDRC. Consequently, the problem now is to determine which classes of digraphs are reconstructible. Chinn[1] proves the following result for graphs :

Theorem A[1]—Let G_i, $i = 1$ to n, be the point-deleted subgraphs of a graph G. If each point-deleted subgraph of one of the G_i, say G_1, occurs exactly once as an induced subgraph of any G_i, $i \neq 1$, then G is reconstructible.

We prove that a related class of digraphs is reconstructible and, from this, reconstruct a larger class of graphs then that covered by Chinn's result.

Let D be a digraph with points $v_1, v_2, \ldots, v_n$, $n > 2$, and assume that v_1 can be located (position of v_1 can be deduced from the information furnished by $D - v_i$, $i = 1$ to n) in $D - v_i$ for each $i > 1$. Let B be the set of degree triples of the point v_1 in $D - v_i$, $i = 2, \ldots, n$. If B has only one element and it is $(n-2, 0, 0)$ $((0, n-2, 0)$ or $(0, 0, n-2))$, then the degree triple of v_1 in D is $(n-1, 0, 0)$ $((0, n-1, 0)$ or $(0, 0, n-1)$, respectively). Otherwise, (k, m, n) is the degree triple of v_1 in D where k, m, and n are, respectively, the maximum values of the first, second and third coordinates among the triples in B. Such an argument works in the case of graphs also. This observation is used in the proof of Theorem 1 below.

2. Reconstruction

Theorem 1 – If D is a digraph having a point j such that $D - j$ has nonisomorphic point-deleted subdigraphs and j can be located (position of point j can be deduced) in each point-deleted subdigraph of D except $D - j$, then D is reconstructible.

Proof : Let $1, 2, \ldots, n$ be the points of D and let us denote $D - i$ by D_i. Without loss of generality, let $j = 1$. By hypothesis, the point 1 can be identified in each D_i, $i = 2$ to n. For each i, $i = 2$ to n, there is a unique point in D_1 whose removal gives a digraph isomorphic to $D_i - 1$. This point of D_1 corressponds to the point i in D and label it in D_1 accordingly. Thus D_1 is labeled with $2, \ldots, n$. From the set B of degree triples of the point 1 in $D_2, \ldots, D_n$, we can find the degree triple (k, m, n) of the point 1 in D as given in the previous section.

It is obvious that there is a single arc from 1 to i, there is a single arc from i to 1, there is a symmetric pair of arcs between 1 and i or there is no arc between 1 and i according as the degree triple of the point 1 in D_i is $(k-1, m, n)$, $(k, m-1, n)$, $(k, m, n-1)$ or (k, m, n). Thus the arc (s) between 1 and any other point i is known. Since the arc (s) between any two points i and j, $1 \neq i \neq j \neq 1$ in D is same as that in the labeled digraph D_1, D is known.

The digraph D in Figure 1 satisfies the hypothesis of Theorem 1 with v in the place of point j as follows :

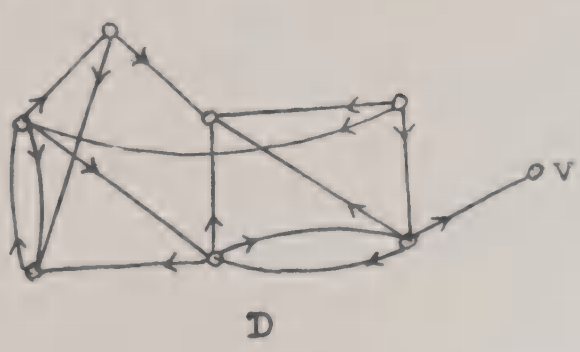

Fig. 1

Suppose the collection S of point-deleted subdigraphs of D are given. In S, there is a member, say D_1 (D_1 corresponds to $D - v$), having distinct point-deleted subdigraphs. From S, we can deduce that the point, say 1, whose deletion gives D_1 is incident with only one arc in D. Hence in any member of S except D_1, point 1 will be incident with at most one arc. However, each member of S except D_1 has exactly one such point and this point must be the point 1. Thus in each member of S except D_1, the point 1 can be located.

Corollary 1 If G is a graph having a point j such that $G - j$ has nonisomorphic point-deleted subgraphs and j can be located (position of point j can be deduced) in each point-deleted subgraph of G except $G - j$, then G is reconstructible.

This is only the graph version of Theorem 1.

Corollary 2—Let D_i, $i = 1$ to n be the point-deleted subdigraphs of a digraph D. If each point-deleted subdigraph of one of the D_i, say D_1 occurs exactly once as an induced subdigraph of any D_i, $i \neq 1$, then D is reconstructible.

PROOF: Since D_i are the point-deleted subdigraphs of a digraph, points of each D_i can be labeled with $\{1, 2, \ldots, n\} - \{i\}$ such that $D_i - j \cong D_j - i$ for each $i \neq j$. If $D_1 - i \cong D_1 - j, i \neq j$, then the point-deleted subdigraph $D_1 - i$ of D_1 is isomorphic to the subdigraphs $D_i - 1$ and $D_j - 1$ of D_i and D_j respectively, contradicting the hypothesis. Hence the point-deleted subdigraphs of D_1 are distinct. Now forget the labeling assumed above. Take any point-deleted subdigraph H of D_1. H can be obtained by deleting a point corresponding to point 1 of D from D_j for some $j > 1$. Because of the hypothesis, there is a unique D_t, $t \neq 1$ and a unique point w in D_t such that $D_t - w$ is isomorphic to H. Clearly this point w in D_t is the point 1 of D. In this way, each point-deleted subdigraph of D_1 can be used to locate the point 1 in a D_i, $i \neq 1$. Since D_i and D_1 have a point deleted subdigraph in common for each $i > 1$, we are able to locate the point 1 in D_i for each $i > 1$. Hence D satisfies the hypothesis of Theorem 1. Hence D is reconstructible.

Theorem A is simply the graph version of the above corollary.

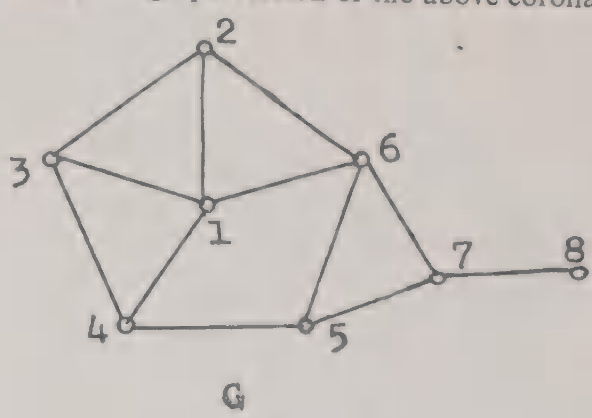

FIG. 2.

The proof of Corollary 2 shows that any digraph (graph) that satisfies the hypothesis of Corollary 2 (Theorem A) satisfies the hypothesis of Theorem 1 (Corollary 1). But the converse is not true. The graph G in Fig. 2 illustrates this. In G, G_8 (G_i denotes $G - i$) has nonisomorphic point-deleted subgraphs. Also, since 8 is a point of degree one in G, it will occur as a point of degree at most one in G_j, $j \neq 8$. However, each G_j, $j \neq 8$, has a unique point with degree at most one, and this point must be the point 8. Thus G satisfies the hypothesis of Corollary 1. But

$$G_1 - 2 \cong G_2 - 1 \cong G_4 - 2$$

$$G_2 - 7 \cong G_3 - 7 \cong G_7 - 2$$

and

$$G_6 - 8 \cong G_5 - 4 \cong G_8 - 6$$

so that there does not exist a G_i such that each point-deleted subgraph of G_i occur exactly once as an induced subgraph of any G_j, $j \neq i$. Hence G does not satisfy the hypothesis of Theorem A. Thus Corollary 1 reconstructs a larger class than that covered by Chinn's result.

Corollary 3—A graph (digraph) G with one point labeled is reconstructible if the subgraph (subdigraph) of G obtained by deleting the labeled point has distinct point-deleted subgraphs (subdigraphs).

The proof is obvious as the hypothesis of Theorem 1 holds.

Acknowledgememt

We are thankful to the referee for many valuable comments.

References

1. P. Z. Chinn, *Recent Trends in Graph Theory*. Lecture Notes in Maths. Vol. 186, Springer-Verlag, Berlin 1971, pp. 71–73.
2. F. Harary, *Graph Theory*, Addison-Wesly Reading, Mass, 1969.
3. S. Ramachandran, *J. Comb. Theory*, (B) **31** (1981), 143–49.
4. S. Ramachandran, *Discrete Math*. **46** (1983) 279–94.
5. P. K. Stockmeyer, *J. Comb. Theory* (B) **31** (1981), 232–39.

India J. pure appl. Math., **20** (8) : 786–789, August 1989

VARIANTS OF HOPFICITY IN MODULES

B. M. PANDEYA

Department of Applied Mathematics, I. T., B. H. U., Varanasi 221005

S. A. PARAMHANS

Department of Mathematics, Faculty of Science, B. H. U., Varanasi 221005

AND

S. P. KOIRALA**

Department of Applied Mathematics, I. T., B. H. U., Varanasi 221005

(*Received* 28 *March* 1988; *after revision* 27 *December* 1988)

It has been shown here that a direct sum of Hopfian modules need not be Hopfian. A class of modules whose strong Hopficity is preserved under taking injective hulls, has been provided. Also a characterization has been obtained for the super Hopficity of a quasi-injective module.

INTRODUCTION

As has been noted by Hiremath[1], the concept of Hopfian groups was introduced by G. Baumslag in 1943. It was generalized to Hopfian rings and modules by Hiremath. In fact, the study of endomorphism rings of various rings and modules has been a topic of keen interest since the end of the nineteen sixties when injectivity and its variants began to flourish. Answering several questions raised by Hiremath, it is shown here that a direct sum of Hopfian modules need not be Hopfian. A class of modules whose strong Hopficity is preserved under taking injective hulls, has also been provided (Proposition 4). Lastly, a characterization has been obtained for the super-Hopficity of a quasi-injective module (Theorem 6).

PRELIMINARIES

Throughout this paper, the basic ring R is associative with 1 and the R-modules M are left unitary. M is said to be uniform if it is essential extension of each of its nonzero submodules. The injective hull of a module M is denoted as $\hat{M}$.

An R module M is said to be Hopfian if every ependomorphism of M is an automorphism of M. Similarly, a ring R is Hopfian if every ring homomorphism of R onto R is an automorphism. Tiwary and Pandeya[2] have introduced the *-property as

***Current address* : Department of Mathematics, Kathamandu University, Nepal.

follows : M is said to satisfy* if every non-zero endomorphism of M is a monomorphism. This property implies Hopficity. But the converse is not true. Thus, we call the *-property "strong Hopficity". Similarly we call M super-Hopfian if its endomorphism ring is a division ring. This property implies strong Hopficity but the converse is not true. These concepts are illustrated by the following examples

(i) Z_4 as a Z-module being Noetherian, is Hopfian but it is not strongly Hopfian.

(ii) Z, considered as module over itself, is cyclic torsionfree and therefore strongly Hopfian but it is not super-Hopfian since $\mathrm{Hom}_Z(Z, Z) \cong Z$ which is not a division ring (cf. Proposition 4).

(iii) Any simple module e.g. Z_2 is a super Hopfian Z-module.

Note 1 : Hiremath[1] has asked whether the Hopficity of a ring R implies the Hopficity of the polynomial ring $R[X]$ in an indeterminate X. We note here that the class of Noetherian rings among the Hopfian ones obviously enjoy this property since R is Noetherian implies $R[X]$ is Noetherian[4] (Theorem 1, p. 201) and hence it is Hopfian.

Further, Hiremath has raised the question whether a direct sum (even finite) of Hopfian modules is Hopfian. It is pointed out here that this is not true at least in the case of infinite sums.

Example 2—The Z-module $\underset{C}{\oplus} Z$ is Hopfian if C is finite. But if C is infinite it is not Hopfian according to Hiremath[1], Proposition 12 and Theorem 14. But each Z is strongly Hopfian and so Hopfian.

This answers part of Hiremath's question in the line just before Remark 7 on p. 897 of Hiremath[1].

Another problem posed by Hiremath[1] (Remark 7) is as follows :

Is Hopficity a hereditary property? Here we note that strong Hopficity is inherited in the case of a quasi-injective module.

Remark 3 : Tiwary and Pandeya[2] (Proposition 2.6) have established that if M is a quasi injective module then the $1-1$-ness of all the nonzero endomorphisms in $\mathrm{Hom}(M, M)$ forces the $1-1$ ness of all the nonzero endomorphisms of any non-zero submodule N of M.

As is noted by Hiremath[1] (Remark 10), the Hopficity of a module does not guarantee the Hopficity of its injective hull. However, we give below a class of strongly Hopfian modules whose injective hulls are also strongly Hopfian.

Proposition 4—A cyclic torsion-free module and its injective hull are strongly Hopfian.

PROOF : Take a nonzero $f \in \mathrm{Hom}\,(M, M)$ and a nonzero element $x \in \ker f$. Then $x = ry$ for some $r \in R$ and $f(x) = r f(y) = 0$. Because $f(y) \neq 0$, $r = 0$ showing that f is monic. Hence M is strongly Hopfian.

Let f be a nonzero endomorphism in $\mathrm{Hom}\,(\hat{M}, \hat{M})$. Suppose $\ker f \neq 0$. Then $\mathrm{Ker}\, f \cap M \neq 0$. So there will exist some nonzero $x \in M$ such that $f(x) = 0$. Since $M = Ry$ this will give $f(ry) = 0$ where $r \in R$. So $rf(y) = 0$. But the torsion-freeness of M gives $f(y) = 0$, a contradiction since $f(y) \neq 0$ as $\hat{M}$ is also the rational extension of M. Hence $\ker f = 0$ and consequently M is strongly Hopfian.

Lastly, we give a characterization for the super-Hopficity of a quasi-injective module. First we have the following :

Lemma 5—If M is quasi-injective and $\hat{M}$ is strongly Hopfian, then M is super-Hopfian.

PROOF : Let $K = \mathrm{Hom}\,(\hat{M}, \hat{M})$ and $D = \mathrm{Hom}\,(M, M)$. Take a nonzero $f \in D$. Due to the injectivity of $\hat{M}$ there exists a nonzero $k \in K$ with $f(M) = k(M)$. Because k is monic, so is f, whence we can define a map $g \in \mathrm{Hom}_R\,(f(M), M)$ by $g(f(m)) = m$. Because M is quasi-injective there exists an $h \in \mathrm{Hom}_R\,(M, M)$ such that $h \mid f(M) = g$. Clearly $hf = 1$ and so D is a division ring.

Theorem 6—Let M be a quasi-injective module. Consider the following conditions :

(i) $\hat{M}$ is super-Hopfian.

(ii) M is super-Hopfian.

(iii) M is uniform and $x, y \in M$, $(0 : x) > (0 : y)$ implies $x = 0$.

Then i) implies ii) whereas ii) and iii) are equivalent. Moreover, if the rational completion of M coincides with $\hat{M}$ then ii) implies i).

PROOF : Let $K = \mathrm{Hom}\,(\hat{M}, \hat{M})$ and $D = \mathrm{Hom}\,(M, M)$.

(i) implies (ii) : This follows from Lemma 5.

(ii) implies (i) : Suppose that the rational completion of M coincides with its injective hull. Let D be a division ring and $k \in K$. Under the given conditions, one-oneness of k is clear from Tiwary and Pandeya[2] (Theorem 1.5).

Since $\hat{M}$ is injective, the exact sequence $0 \to \hat{M} \to k(\hat{M})$ splits and so $k(\hat{M}) = \hat{M}$, whence k is an isomorphism.

(ii) implies (iii) and conversely : This follows from Wong[3].

Corollary—If we take M to be nonsingular in the above Theorem then i), ii) and iii) are all equivalent.

Acknowledgement

The authors are thankful to the referee for his valuable suggestions, in particular, for the Example 2 and the statement alongwith proof of Lemma 5. The second author gratefully acknowledges financial support by U. G. C. India.

References

1. V. A. Hiremath, *Indian J. pure appl. Math.* **17** (1986), 895-900.
2. A. K. Tiwary and B. M. Pandeya, *Proceedings of Seminar on Algebra and its Application* (Edited by H. L. Manocha and J. B. Srivastava), Marcel Dekker Lecture Notes No. 91 (1984), 199-203.
3. E. T. Wong, *Canad. Math. Bull.* **13** (1) (1970), 260-61.
4. O. Zariski and P. Samuel, *Commutative Algebra*, Vol. I. Von Nostrand Co. Inc. New York, 1958.

Indian J. pure appl. Math., **20** (8) : 790-798, August 1989

ON hv-RECURRENT FINSLER CONNECTION

B. N. Prasad

Department of Mathematics, St. Andrew's College, Gorakhpur

AND

Lalji Srivastava

Department of Mathematics, M. G. Degree College, Gorakhpur

(*Received* 14 *September* 1987; *after revision* 25 *December* 1988)

The purpose of the present paper is to introduce a Finsler connection which is neither h-metrical nor v-metrical but it is recurrent with respect to both h and v-covariant derivatives. Such a Finsler connection will be called an hv-recurrent Finsler connection.

1. Introduction

Cartan[1] published his monograph "Les espaces de Finsler" and fixed his method to define a notion of connection in the geometry of Finsler space. His method was put in order by Matsumoto[6] and determined uniquely the Cartan connection by assuming four elegent axioms :

(1) The connection is metrical.

(2) The deflection tensor field vanishes.

(3) The torsion tensor field T vanishes.

(4) The torsion tensor field S^1 vanishes.

Hashiguchi[3] replaced the condition 2 by some weaker condition and determined a Finsler connection with the given deflection tensor field. He[4] also determined uniquely a Finsler connection, by replacing the condition 3. In almost all these works it has been assumed that the connection is metrical so that the covariant differentiations commute with the raising and lowering of indices.

Recently Prasad *et al.*[11] have introduced a Finsler connection with respect to which the metric tensor is h-recurrent. Such a Finsler connection has been called an h-recurrent Finsler connection. While introducing an h-recurrent Finsler connection, it has been assumed that the v-covariant derivative of the metric tensor vanishes and the torsion tensor fields T and S^1 also vanish. The notion of hv-recurrent Finsler conneetion has also been studied by Ghinea[2] from another standpoint.

A Finsler manifold (F^n, L) of dimension n is a manifold F^n associated with a fundamental function $L(x, y)$ where $x\ (= x^i)$ denote the positional variables of F^n and $y\ (= y^i)$ denote the components of a tangent vector with respect to x^i. The metric tensor of (F^n, L) is given by $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$ where $\dot{\partial}_i = \partial/\partial y^i$.

A Finsler connection of (F^n, L) is tried $\left(F^i_{jk},\ N^i_k,\ C^i_{jk} \right)$ of a V-connection F^i_{jk}, a non-linear connection N^i_k and a vertical connection C^i_{jk} (Matsumoto[7]). If a Finsler connection is given, the h-and v-covariant derivatives of any tensor field V^i_j are defined as

$$V^i_{j|k} = d_k V^i_j + V^m_j F^i_{mk} - V^i_m F^m_{jk}$$

$$V^i_j \Big|_k = \dot{\partial}_k V^i_j + V^m_j C^i_{mk} - V^i_m C^m_{jk}$$

where

$$d_k = \partial_k - N^m_k \dot{\partial}_m, \partial_k = \partial/\partial x^k.$$

2. hv-Recurrent Finsler Connections

Let a_k be the components of a (0) p-homogeneous vector field and b_k be the components of a (-1) p-homogeneous vector field. Then a Finsler connection $F(a, b) = \{F^i_{jk}(a, b), N^i_k(a, b), C^i_{jk}(a, b)\}$ will be called hv-recurrent if the h-and v-covariant derivatives of the metric tensor g_{ij} with respect to $F(a, b)$ are recurrent, i. e , $g_{ij|k} = a_k g_{ij}$ and $g_{ij} \mid_k = b_k g_{ij}$. In particular hv-recurrent Finsler connections $F(a, 0)$ and $F(0, b)$ will be called h and v-recurrent respectively. The quantities with respect to hv-, h-and v-recurrent Finsler connections will be denoted by putting (a, b), (a) and (b) respectively. The quantities without any parenthesis will correspond to the quantities with respect to the Cartan connection $C\Gamma$. To avoid confusions we use the h and v-covariant derivatives with respect to $C\Gamma$ by $|\ k$ and $|\ k$ while these covariant derivatives with respect to any hv-recurrent Finsler connection will be denoted by $\|\ k$ and $\|\ k$. To determine such an hv-recurrent Finsler connection we have the following.

Theorem 2.1—Given covariant vector fields a_k and b_k, there exists a unique Finsler connection $F(a, b)$ satisfying the axioms

$$g_{ij\|k} = a_k g_{ij},\ g_{ij}\|_k = b_k g_{ij},\ N^i_k(a, b) = y^j F^i_{jk}(a, b)$$

$$F^i_{jk}(a, b) = F^i_{kj}(a, b),\ C^i_{jk}(a, b) = C^i_{kj}(a, b).$$

The coefficients are given by

$$F^{i}_{jk}(a, b) = \Gamma^{*i}_{jk} + Q^{i}_{jk} \qquad \ldots(2.1)$$

$$N^{i}_{k}(a, b) = G^{i}_{k} + T^{i}_{k} \qquad \ldots(2.2)$$

$$C^{i}_{jk}(a, b) = C^{i}_{jk} + \sigma^{i}_{jk} \qquad \ldots(2.3)$$

where

$$C\Gamma = \left(\Gamma^{*i}_{jk},\ G^{i}_{k},\ C^{i}_{jk}\right)$$

$$Q^{i}_{jk} = \tfrac{1}{2}\Big\{a_0\, C^{i}_{jk} + L^2\Big(C^{i}_{jm}\ C^{m}_{k} + C^{i}_{km}\ C^{m}_{j} - C^{m}_{jk}\ C^{i}_{m}\Big)$$
$$-\Big(C^{i}_{j}\, y_k + C^{i}_{k}\, y_j - C_{jk}\, y^i\Big) - \Big(a_k\, \delta^{i}_{j} + a_j \delta^{i}_{k} - a^i g_{jk}\Big)\Big\} \qquad \ldots(2.4)$$

$$T^{i}_{k} = \tfrac{1}{2}\Big(a^i\, y_k - a_k\, y^i - a_0\, \delta^{i}_{k}\, L^2\, C^{i}_{k}\Big) \qquad \ldots(2.5)$$

$$\sigma^{i}_{jk} = \tfrac{1}{2}\Big(b^i g_{jk} - b_j\, \delta^{i}_{k} - b_k \delta^{i}_{j}\Big) \qquad \ldots(2.6)$$

$$C^{i}_{k} = a^m\, C^{i}_{km},\ b^i = g^{im}\, b_m$$

and 0 denotes the contraction with y^i.

In the following, we establish the relation between the torsion R', P' and cruvature tensors corresponding to $F(a, b)$ R^2, P^2, S^2 determined in Theorem (2.1) and $C\Gamma$. (k) denotes the h-covariant derivative with respect to the Berwald connection $B\,\Gamma = (\dot{\partial}_k\, G^{i}_{j}\ ,\ G^{i}_{k}\ , 0)$.

By direct calculations based on their definitions, we get

$$R^{i}_{jk}\ (a, b) = R^{i}_{jk} + T^{i}_{j(k)} - T^{i}_{k(j)} + T^{m}_{j}\ \dot{\partial}_m\, T^{i}_{k} - T^{m}_{k}\, \dot{\partial}_m\ T^{i}_{j} \qquad \ldots(2.7)$$

$$P^{i}_{jk}\,(a, b) = P^{i}_{jk} + \dot{\partial}_k\ T^{i}_{j} - Q^{i}_{kj} \qquad \ldots(2.8)$$

$$R^{i}_{hjk}\ (a, b) = R^{i}_{hjk} + Q^{i}_{hj|k} - Q^{i}_{hk|j} - T^{m}_{k}\ \dot{\partial}_m\, \Gamma^{*i}_{hj} - T^{m}_{k}\, \dot{\partial}_m\, Q^{i}_{hj}$$
$$+ T^{m}_{j}\ \dot{\partial}_m\ \Gamma^{*i}_{hk} + T^{m}_{j}\, \dot{\partial}_m\, Q^{i}_{hk} + Q^{m}_{hj}\ Q^{i}_{mk} - Q^{m}_{hk}\ Q^{i}_{mj}$$

(*equation continued on p. 793*)

$$+ \sigma^{i}_{hm} \; R^{m}_{jk} + \left(C^{i}_{hm} + \sigma^{i}_{hm} \right) \left(T^{m}_{j(k)} - T^{m}_{k(j)} + T^{r}_{j} \, \partial_r \; T^{m}_{k} \right.$$

$$\left. - T^{r}_{k} \, \dot{\partial}_r \; T^{m}_{j} \right) \qquad \ldots(2.9)$$

$$P^{i}_{hjk}\,(a, b) = P^{i}_{hjk} + T^{m}_{j} \, \dot{\partial}_m \left(C^{i}_{hk} + \sigma^{k}_{hk} \right)$$

$$+ \left(C^{i}_{hm} + \sigma^{i}_{hm} \right) \dot{\partial}_k \, T^{m}_{j} + Q^{i}_{hj|k}$$

$$- \sigma^{i}_{hk|j} + \sigma^{i}_{hm} \; P^{m}_{jk} + Q^{i}_{hm} \; C^{m}_{jk} - \sigma^{m}_{hk} \; Q^{i}_{mj} + \sigma^{i}_{mk} \; Q^{m}_{hj} \qquad \ldots(2.10)$$

$$S^{i}_{hjk}\,(a, b) = S^{i}_{hjk} + \tfrac{1}{2}\, b^m \left\{ C^{i}_{mk}\, g_{hj} + C_{mhk}\, \delta^{i}_{j} - C^{i}_{mj}\, g_{hk} \right.$$

$$\left. - C_{mhj}\, \delta^{i}_{k} \right\} + \tfrac{1}{4}\, b^2 \left(g_{hk} \;\; \delta^{i}_{j} - g_{hj} \;\; \delta^{i}_{k} \right) + \tfrac{1}{4}$$

$$b^i \,(g_{hj}\, b_k - g_{hk}\, b_j) + \tfrac{1}{4}\, b_h \left(b_j \;\; \delta^{i}_{k} - b_k \;\; \delta^{i}_{j} \right)$$

$$+ \dot{\partial}_k \, \sigma^{i}_{hj} - \dot{\partial}_j \, \sigma^{i}_{hk} \qquad \ldots(2.11)$$

where

$$b^2 = b_m \, b^m.$$

3. h-Recurrent Finsler Connections

In this section we consider a particular one $F\,(a)$ of an h-recurrent Finsler connection in which a_k is the unit vector (Putting $a_k = l_k$, $b_k = 0$ in (2.4), (2.5) and (2.6) we get

$$Q^{i}_{jk} = - \tfrac{1}{2} \left(L\, C^{i}_{jk} - l_k \, \delta^{i}_{j} - l_j \, \delta^{i}_{k} + l^i \, g_{jk} \right) \qquad \ldots(3.1)$$

$$T^{i}_{k} = - \tfrac{1}{2}\, L\, \delta^{i}_{k} \, , \; C^{i}_{k} = 0 \qquad \ldots(3.2)$$

$$\sigma^{i}_{jh} = 0. \qquad \ldots(3.3)$$

Thus from (2.1), (2.2) and (2.3) we get

$$F^{i}_{jk}\,(a) = \Gamma^{*i}_{jk} + \tfrac{1}{2} \left(L\, C^{i}_{jk} - l_k \, \delta^{i}_{j} - l_j \, \delta^{i}_{k} + l^i \, g_{jk} \right) \qquad \ldots(3.4)$$

$$N^{i}_{k}\,(a) = G^{i}_{k} - \tfrac{1}{2}\; L \; \delta^{i}_{k} \qquad \ldots(3.5)$$

$$C^i_{jk}(a) = C^i_{jk}\,. \qquad \ldots(3.6)$$

Since $L_{(k)} = 0$, in view of (2.7) and (3.2) we have

$$R^i_{jk}(a) = R^i_{jk} + \tfrac{1}{4}\left(y_j\,\delta^i_k - y_k\,\delta^i_j\right). \qquad \ldots(3.7)$$

By Matsumoto[9] (p. 168) a Finsler space of scalar curvature K is characterized by

$$R^i_{jk}\,y^j = K\left(L^2\,\delta^i_k - y_k\,y^i\right).$$

If K is constant then (F^n, L) is said to be of constant curvature[9] (p. 170). In view of the relation (3.7) we have the following

Theorem 3.1—If the (v) h-torsion tensor $R^i_{jk}(a)$ of an h-recurrent Finsler connection $F(a)$ with respect to $a_k = l_k$ vanishes then (F^n, L) is of constant curvature $(-\tfrac{1}{4})$.

Substituting (3.1) and (3.2) in (2.8) we get

$$P^i_{jk}(a) = P^i_{jk} - \tfrac{1}{2}\left(L\,C^i_{jk} - l_j\,\delta^i_k + l^i\,g_{jk}\right).$$

This relation gives

$$P^i_{0k}(a) = \tfrac{1}{2}\,L\,h^i_k\,,\quad P^0_{jk}(a) = -\tfrac{1}{2}\,L\,h_{jk}$$

$$P^i_{jk}(a) - P^i_{kj}(a) = \tfrac{1}{2}\left(l_j\,\delta^i_k - l_k\,\delta^i_j\right)$$

which give the following

Theorem 3.2—The (h) hv-torsion tensor $P^i_{jk}(a)$ of an h-recurrent Finsler connection $F(a)$ with respect to $a_k = l_k$ never vanishes.

Substitting the values of Q^i_{jk}, T^i_j and σ^i_{jk} from (3.1), (3.2) and (3.3) in the relation (2.9) we get

$$R^i_{hjk}(a) = R^i_{hjk} + \tfrac{1}{2}\,L\left(P^i_{hjk} - P^i_{hkj}\right) + \frac{1}{4}\,L^2\,S^i_{hjk}$$
$$+ \frac{1}{4}\left(\delta^i_k\,g_{hij} - \delta^i_j\,g_{hk}\right).$$

Since

$$P^i_{hjk} - P^i_{hkj} = -\,S^i_{hjk|0}$$

(Matsumoto[9], p. 115), we have the following

Theorem 3.3—If the h-curvature tensor $R^{i}_{hjk}(a)$ of an h-recurrent Finsler connection $F(a)$ with respect to $a_k = l_k$ vanishes and (F^n, L) is of constant curvature $(-\frac{1}{4})$ then

$$S^{i}_{hjk|0} = \tfrac{1}{2} L \ S^{i}_{hjk} \ .$$

To find the relation between the hv-curvature tensors of an h-recurrent Finsler connection $F(a)$ with respect to $a_k = l_k$ and $C\Gamma$, we differentiate (3.1) v-covariantly with respect to $C\Gamma$. Then we have

$$Q^{i}_{hj|k} = \tfrac{1}{2}\left(l_k C^{i}_{hj} + L\, C^{i}_{hj|k} - \frac{1}{L} h_{hk}\, \delta^{i}_{j} - \frac{1}{L} h_{jk}\, \delta^{i}_{h} + \frac{1}{L} h^{i}_{k}\, g_{hj}\right).$$

Substituting (3.1), (3.2) and (3.3) in (2.10) we get

$$P^{i}_{hjk}(a) = P^{i}_{hjk} + \tfrac{1}{2} L\, S^{i}_{hjk} + \tfrac{1}{2}\left(l_i C_{jkh} - l_h C^{i}_{jk}\right) + \frac{1}{2L}\left(h^{i}_{k}\, g_{hj} - h_{hk}\, \delta^{i}_{j} - h_{jk}\, \delta^{i}_{h}\right).$$

Since the vertical connection $C^{i}_{jk}(a)$ of an h-recurrent Finsler connection $F(a)$ is the same as the one C^{i}_{jk} of $C\Gamma$ cf. (3.6)), the v-curvatures of both the connections will be same.

4. v-Recurrent Finsler Connections

A v Recurrent Finsler connection $F(b)$ is a particular hv-recurrent Finsler connection $F(a, b)$ obtained by putting $a_k = 0$. Then we have $T^{i}_{k} = 0$ and $Q^{i}_{jk} = 0$. Thus from (2.1), (2.2) and (2.3) we get

$$F^{i}_{jk}(b) = \Gamma^{*i}_{jk}\ , N^{i}_{k}(b) = G^{i}_{k}\ , C^{i}_{jk}(b) = C^{i}_{jk} + \sigma^{i}_{jk}\ .$$

It is to be noted that our v-recurrent Finsler connection is the generalized Cartan connection $C_\sigma\Gamma$ defined by Hōjō[5]. From (2.7), (2.8), (2.9) and (2.10), it follows that the torsion R^1, P^1 and curvature tensors R^2, P^2 of $F(b)$ are given by

$$R^{i}_{jk}(b) = R^{i}_{jk}\ ,\quad P^{i}_{jk}(b) = P^{i}_{jk} \qquad \text{...(4.1)}$$

$$R^{i}_{hjk}(b) = R^{i}_{hjk} + \tfrac{1}{2}\left(b^{i} R_{hjk} - b_m \delta^{i}_{h} R^{m}_{jk} - b_h R^{i}_{jk} \right) \qquad \ldots(4.2)$$

$$P^{i}_{hjk}(b) = P^{i}_{hjk} - \tfrac{1}{2}\left(b^{i}_{|j} g_{hk} - b_{k|j} \delta^{i}_{h} - b_{h|j} \delta^{i}_{k} \right)$$

$$+ \tfrac{1}{2}\left(b^{i} P_{hjk} - b_m P^{m}_{jk} \delta^{i}_{h} - b_h P^{i}_{jk} \right). \qquad \ldots(4.3)$$

Now if $R^{i}_{hjk} = 0$ then $R^{i}_{jk} = R^{i}_{hjk} y^{h} = 0$. Hence from (4.2) we have $R^{i}_{hjk}(b) = 0$. Conversely, if $R^{i}_{hjk}(b) = 0$ then after contracting (4.2) with y^{h} we get

$$R^{i}_{jk} + \tfrac{1}{2}\left(b^{i} R_{0jk} - b_m y^{i} R^{m}_{jk} - b_0 R^{i}_{jk} \right) = 0. \qquad \ldots(4.4)$$

Again contracting it with y_i we get

$$R_{0jk} = \tfrac{1}{2} L^{2} b_m R^{m}_{jk}. \qquad \ldots(4.5)$$

Transvecting (4.4) with b_i we get

$$(1 - b_0) R^{m}_{jk} b_m + \tfrac{1}{2} b^{2} R_{0jk} = 0. \qquad \ldots(4.6)$$

From (4.5) and (4.6) we get $R_{0jk} = 0 = R^{m}_{jk} b_m$ provided $4(1-b_0) + b^{2} L^{2} \neq 0$. Hence from (4.4) we get $R^{i}_{jk} = 0$ provided $b_0 \neq 2$. Therefore putting $R^{i}_{hjk}(b) = 0$, $R^{i}_{jk} = 0$ in (4.2) we get $R^{i}_{hjk} = 0$. Thus we get the following

Theorem 4.1—If a v-recurrent Finsler connection $F(b)$ satisfies $4(1 - b_0) + b^{2} L^{2} \neq 0$, $b_0 \neq 2$ then $R^{i}_{hjk}(b) = 0$ is equivalent to $R^{i}_{hjk} = 0$.

If $P^{i}_{hjk} = 0$ then $P^{i}_{jk} = P^{i}_{hjk} y^{h} = 0$. Hence from (4.3) we have $P^{i}_{hjk}(b) = 0$ provided $b_{i|k} = 0$. Conversely if $P^{i}_{hjk}(b) = 0 = b_{i|k}$ then we have

$$P^{i}_{hjk} + \tfrac{1}{2}\left(b^{i} P_{hjk} - b_m P^{m}_{jk} \delta^{i}_{h} - b_h P^{i}_{jk} \right) = 0. \qquad \ldots(4.7)$$

Contracting (4.7) with y^{h} and using the fact that $P_{hjk} y^{h} = 0$, we get

$$(2 - b_0) P^{i}_{jk} = b_m P^{m}_{jk} y^{i}. \qquad \ldots(4.8)$$

Again contracting (4.8) with b_i we get $b_m\ P^m_{jk} = 0$ provided $b_0 \neq 1$. Hence (4.8) yields $P^i_{jk} = 0$ if $b_0 \neq 2$. Thus (4.7) gives $P^i_{hjk} = 0$. Hence we have the following.

Theorem 4.2—If a v-recurrent Finsler connection $F(b)$ satisfies $b_{i|k} = 0, b_0 \neq 1,2$ then $P^i_{hjk}(b) = 0$ is equivalent to $P^i_{hjk} = 0$.

Since the vertical connection $C^i_{jk}(a, b)$ of an hv-recurrent Finsler connection $F(a, b)$ is identical to the one $C^i_{jk}(b)$ of the corresponding v-recurrent Finsler connection, $F(b)$ the v-curvature tensor $S^i_{hjk}(b)$ of $F(b)$ is also given by (2.11).

If the recurrence vector b_k is such that $b_k = L^{-1}\ l_k$ then (2.11) is reduced to

$$S^i_{hjk}(b) = S^i_{hjk} - \frac{3}{4L^2}\left(h_{hk}\ h^i_j - h_{hj}\ h^i_k\right) \qquad ...(4.9)$$

A Finsler space of dimension $n \geqslant 4$ is called S_3-like if S^i_{hjk} is of the form[10]

$$L^2\ S^i_{hjk} = S\left(h_{hj}\ h^i_k - h_{hk}\ h^i_j\right)$$

where S is a scalar. In this case the scalar S is a function of position alone[8]. Therefore (4.9) gives the following :

Theorem 4.3—If v-curvature tensor $S^i_{hjk}(b)$ of a v-recurrent Finsler connection $F(b)$ with respect to $b_k = L^{-1}\ l_k$ vanishes then (F^n, L) is S_3-like. In this case $S = -\frac{3}{4}$.

Acknowledgement

The authors are thankful to the referee for the useful suggestions. The second author is grateful to Dr R. S. Sinha for guidance and advice in the preparation of this paper.

References

1. E. Cartan, Les espaces de Finsler. Actualitiés 79. Paris, 1934.
2. I. Ghinea, *Anale Univ. din. Timisoara* **14** (1976), 101-15.
3. M. Hashiguchi, *Rep. Fac. Sci. Kagoshima Univ.* (*Math. Phys. Chem.*) **2** (1969), 29-39.
4. M. Hashiguchi, *J. Korean Math. Soc.* 12 (1975), 51-61.
5. S. Hōjō, *Tensor, N. S.* 35 (1981), 333-44.
6. M. Matsumoto, *Tensor, N. S.* **17** (1966), 217-26.

7. M. Matsumoto, *Publ. of the Study Group of Geometry* 5. Dept. Math. Okayam Univ., 1970, 120 p.
8. M. Matsumoto, *Tensor, N. S.* **22** (1971), 201-204.
9. M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*. Kaiseisha Press Otsu, Japan, (1986).
10. M. Matsumoto and C. Shibata, *J. Math. Kyoto Univ.* **19** (1979), 301-14.
11. B. N. Prasad, H. S. Shukla and D. D. Singh, *Pub. Math. Debrecen*. **35** 1988.

Indan J. pure appl. Math., **20** (8) : 799-803, August 1989

HYPERSURFACES WITH (f, g, u, v, λ)—STRUCTURE OF AN AFFINELY COSYMPLETIC MANIFOLD

DHRUWA NARAIN

Department of Mathematics, University of Gorakhpur, Gorakhpur 273001

(*Received* 4 *September* 1986; *after final revision* 9 *February* 1989)

In this paper we shall show the existence of quartic structure of a hypersurface in an almost contact manifold. Applying cosymplectic conditions on almost contact manifold, we also study totally umbilical, geodesic and the existence of parallel vector fields in the noninvariant hypersurface with (f, g, u, v, λ)—structure and some properties of normal (f, g, u, v, λ)—structure.

1. QUARTIC STRUCTURE

Let M be $(2n+1)$ dimensional almost contact manifold with $(1,1)$ type tensor ϕ, a fundamental vector field E and a contact form η. Let us consider a $2n$-dimensional manifold P embedded in M with embedding $i : P \to M$.

Let us choose on affine normal N on P in such a way that ϕN is always tangent to hypersurface and satisfy following linear transformations :

$$\phi i_* X = i_* fX + u(X) N \qquad \ldots(1.1)$$

$$\phi N = -i_* U \qquad \ldots(1.2)$$

$$E = l_* V + \lambda N \qquad \ldots(1.3)$$

$$\eta(i_* X) = v(X) \qquad \ldots(1.4)$$

where f is a $(1,1)$ type tensor; U, V are vector field, u, v are 1-forms and λ a C^∞-function. If $u \neq 0$ P is called a noninvariant hypersurface of M.

From (1.1), (1 2), (1.3), (1.4) and using properties of almost contact structure (ϕ, E, η), we have the following induced structure on P

$$\left.\begin{aligned} & f^2 X = -X + u(X) U + v(X) V \\ & u(fX) = \lambda v(X), \; v(fX) = -\eta(N) u(X), \\ & fu = -\eta(N) V, \quad f(V) = \lambda U, \\ & u(U) = 1 - \lambda\eta(N), u(V) = 0, \\ & v(U) = 0, v(V) = (1 - \lambda\eta(N)). \end{aligned}\right\} \qquad \ldots(1.5)$$

If the vector fields E and N are distinct affine normals on P, then P has a quartic structure

$$f^4 + (1 + \lambda\eta(N)) f^2 + \lambda\eta(N) I = 0. \quad \ldots(1.6)$$

The above equation may be factorized as

$$(f^2 + \lambda\eta(N) I)(f^2 + I) = 0. \quad \ldots(1.7)$$

If we put $\eta(N) = \lambda$ in (1.5), we have

$$\begin{aligned} & f^2 = -I + u \otimes U + v \otimes V \\ & fU = -\lambda V, fV = \lambda U \\ & u \circ f = \lambda v, v \circ f = -\lambda u \\ & u(U) = 1 - \lambda^2, u(V) = 0 \\ & v(U) = 0, v(V) = 1 - \lambda^2 \end{aligned} \quad \ldots(1.8)$$

which is an (f, U, V, u, v, λ) – structure[4]. Now if we introduce a metric g on the (f, U, V, u, v, λ) – structure, such that

$$\left.\begin{aligned} & g(U, X) = u(X), g(V, X) = v(X) \\ & g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y). \end{aligned}\right\} \quad \ldots(1.9)$$

Then above structure reduces to an (f, g, u, v, λ) – structure.

2. Hypersurfaces of an Affinely Cosymplectic Manifolds

If $M(\phi, E, \eta)$ is an affinely cosymplectic manifold, i.e. $\nabla\phi = 0$ and $\nabla\eta = 0$, where ∇ denotes the covariant differentiation on M. Since $\nabla\eta = 0$ implies that $\nabla E = 0$, i.e. the vector field E is parallel with respect to ∇. Let D be the induced connection on the hypersurface P of the affine connection ∇. Now using Gauss and Weingarten's equations

$$\nabla_{i_* X} \overset{*}{i} Y = i_* D_X Y + h(X, Y) N \quad \ldots(2.1)$$

and

$$\nabla_{i_* X} N = i_* HX + \omega(X) N \quad \ldots(2.2)$$

where h and H are the second fundamental tensors of type (0,2) and (1,1) and ω is a 1-form.

Now differentiating (1.1), (1.2), (1.3), (1.4) covariantly and using $\nabla\phi = 0$, $\nabla\eta = 0$, $\nabla E = 0$ and (2.1), (2.2) and reusing (1.1), (1.2), (1.3), (1.4), we get

(a) $(D_X f)(Y) = u(Y) HX - h(X, Y) U$

(b) $D_X V = \lambda HX, \; D_X U = fHX + \omega(X) U$

(c) $(D_X v)(Y) = \lambda h(X, Y)$, $(D_X u)(Y) = -h(X, fY) - \omega(X) u(Y)$

(d) $h(X, V) = -X\lambda - \lambda\omega(X)$. ...(2.3)

Theorem 2.1—If hypersurface P is endowed with (f, g, u, v, λ)—structure and if it is an affinely umbilical hypersurface of an affinely cosymplectic manifold, it is totally geodesic, iff

$$\omega = -d(\log \lambda).$$

PROOF : Since P is affinely umbilical, putting $H = \mu I$, (2.3)d yields

$$-X\lambda - \lambda\omega(X) = v(HX) = \mu v(X). \quad ...(2.4)$$

Thus if $\mu = 0$, P is totally geodesic from (2.4), we get

$$-X\lambda - \lambda\omega(X) = 0.$$

Hence

$$\omega(X) = -\frac{X\lambda}{\lambda} = -d(\log \lambda)(X)\,[df(X) = Xf]$$

i.e.

$$\omega = -d(\log \lambda).$$

Conversely, if $\omega = -d(\log \lambda)$, we get $-X\lambda - \lambda\omega(X) = 0$, which implies that $\mu=0$, since $v(X) \neq 0$.

Theorem 2.2—Let P be a noninvariant hypersurface of an affinely cosymplectic manifold with (f, g, u, v, λ)—structure. Then if the linear transformation field f is a parallel field, we have

$$(1 - \lambda^2)^2 h(X, Y) = \nu u(X) u(Y) \quad ...(2.5)$$

$$\omega = -d(\log \lambda) \quad ..(2.6)$$

where $\nu = h(U, U)$, i.e. P is cylindrical hypersurface.

PROOF : Since the linear transformation field f is a parallel field, from (2.3) a, we have

$$u(Y) u(HX) = (1 - \lambda^2) h(X, Y). \quad ...(2.7)$$

Since h is symmetric, $u(Y) u(HX) = u(X) u(HY)$ and putting $Y = U$, we get

$$\nu u(X) = (1 - \lambda^2) u(HX). \quad ...(2.8)$$

Now from (2.7) and (2.8), we get (2 5), which shows that P is cylindrical. Further putting $Y = V$ in (2.3) a, we get $h(X, V) = 0$, which from (2.3) d is $X\lambda = -\lambda\omega(X)$, implies $\omega(X) = -\dfrac{X\lambda}{\lambda} = -d(\log \lambda)\ X$.

Theorem 2.3—Let P be a noninvariant hypersurface of an affinely cosymplectic manifold with (f, g, u, v, λ) – structure. Then if P is totally geodesic, f is parallel with respect to induced connection.

PROOF : Since P is totally geodesic, $h = 0$, implies $H = 0$, which yields $Df = 0$, i.e. f is a parallel vector field.

Theorem 2.4—The noninvariant totally geodesic hypersurface P with (f, g, u, v, λ) – structure of an almost cosymplectic manifold, the vector fields U, V are parallel vector fields, if λ is constant.

PROOF : Since P is totally geodesic, we have $\omega = -d(\log \lambda)$. If λ is constant $\omega(X) = 0$. Now from (2.3 b), we get $DU = 0$ and $DV = 0$.

3. NORMAL (f, g, u, v, λ) – STRUCTURE

An (f, g, u, v, λ) – structure is said to be normal if the torsion tensor S of f satisfies

$$S(X, Y) = N(X, Y) + du(X, Y)\, U + dv(X, Y)\, V = 0 \qquad \ldots(3.1)$$

where N is the Nijenhuis tensor, and

$$du(X, Y) = (D_X u)(Y) - (D_Y u)(X)$$

$$dv(X, Y) = (D_X v)(Y) - (D_Y v)(X).$$

Theorem 3.1—The noninvariant hypersurface P with (f, g, u, v, λ) – structure of affinely cosymplectic manifold is normal, if f commutues with H and $\omega = \alpha' u$.

PROOF : The Nijenhuis tensor N of f is given by

$$N(X, Y) = (D_{fX} f)(Y) - (D_{fY} f)(X) - f(D_X f)(Y)$$
$$+ f(D_Y f)(X). \qquad \ldots(3.2)$$

Using (2.3 a), we get

$$N(X, Y) = u(Y)(HfX - fHX) + u(X)(fHY - HfY)$$
$$+ (g(HfY, X) - g(HfX, Y))\, U.$$

Now from (2.3 c), $dv = 0$

and

$$du(X, Y) = -h(X, fY) - \omega(X)\, u(Y) + h(Y, fX) + \omega(Y)\, u(X).$$

Now putting $fH = Hf$ and $\omega = \alpha' u$ in (3.1), yields $S = 0$.

Theorem 3.2—If the hypersurface P with (f, g, u, v, λ) – structure is normal, we have

$$\eta^a (N(X, Y) + (1 - \lambda^2)\, d\,\eta^a (X, Y) = 0 \qquad \ldots(3.3)$$

$$\overline{\overline{N(X, Y)}} + \lambda^2 N(X, Y) = 0 \qquad \text{...(3.4)}$$

$$(1 - \lambda^2)(d\eta^a(X, \bar{Y}) + d\eta^a(\bar{X}, Y)) + \eta^a(N(X, \bar{Y}) + N(\bar{X}, Y)) = 0 \qquad \text{...(3.5)}$$

where

$$\bar{X} = fX \text{ and } a = 1,2;\ \eta^1 = u,\ \eta^2 = v.$$

PROOF : Operating (3.1) by η^a, we get (3.3). Barring (3.1) twice and using (1.8), we get

$$\overline{\overline{N(X, Y)}} - \lambda^2 \{du(X, Y)\,U + dv(X, Y)\,V\} = 0. \qquad \text{...(3.6)}$$

Now multiplying (3.1) by λ^2 and adding with (3.6) we get (3.4). Barring X, Y in (3.1) respectively and adding them, we get

$$N(\bar{X}, Y) + N(X, \bar{Y}) + (d\eta^a(\bar{X}, Y) + d\eta^a(X, \bar{Y}))\,E_a = 0.$$

Operating the above by η^a, we get (3.5).

REFERENCES

1. S. I. Goldberg, *J. Diff. Geom.* **5** (1971), 211–27.
2. R. S. Mishra, *Structure on a Differentiable Manifold and their Applications*. Chandrama Prakashan, 1984.
3. B. B. Sinha, and Dhruwa Narain, *Indian J. pure appl. Math.*, **9** (1978), 753–60.
4. K. Yano and M. Okumura, *Kōdai Math. Sem. Rep.* **22** (1970), 401–23.

Indian J. pure appl. Math., **20** (8) . 804–811, August 1989

O-DISTRIBUTIVE POSETS

Y. S. PAWAR AND V. B. DHAMKE

Department of Mathematics, Shivaji University, Kolhapur 416004

(*Received* 7 *December* 1989)

In this paper O-distributivity in a partially ordered set (poset) is defined. Some equivalent formulations for O-distributivity, in poset, are obtained. It is shown that o-distributive poset is a generalization of pseudocomplemented poset. Mainly, we prove the following :

Theorem—For a O-distributive poset P, the set of all annihilator ideals, $A(P)$, is a Boolean algebra.

1. INTRODUCTION

Venkatanarasimhan[5] has defined pseudocomplemented partially ordered sets (posets). He proved that a poset P with O is pseudocomplemented if and only if $(a]^*$ is a principal ideal for every a in P. But it is observed that for $(a]^*$ to be an ideal (in the sense of Venkatanarasimhan[5], it is necessary and sufficient that, P is O-distributive. Also the purpose of this paper is to extend some of the results of Pawar[3] to partially ordered sets. Note that the definition of O-distributive semilattices given by Pawar[3] is different from that given by Varlet[4].

In section 1 we collect some known results and definitions which are used in subsequent sections. Section 2 deals with the definition, examples and several properties of O-distributive poset. In the concluding section annihilator ideals in a O-distributive poset are studied in detail.

2. PRELIMINARIES

P denotes a partially ordered set with the ordering relation $\leqslant$. For a finite set $A = \{a_1, a_2, \ldots a_n\}$ the least upper bound (l.u.b.) and the greatest lower bound (g.e.b.) of A are denoted by $a_1 \vee a_2 \vee \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge \ldots \wedge a_n$ respectively. The least and the greatest elements of a poset, when they exist, are denoted by O and 1 respectively. A non-null subset A of P is called as a semi-ideal if $a \in A$, $b \leqslant a \Rightarrow b \in A$. A semi-ideal A of P is called as an ideal if the least upper bound of any finite number of elements of A, whenever it exists, belongs to A. This definition of an ideal in a poset given by Venkatanarasimhan[5] is different from that introduced by Frink[2]. Set inclusion, set intersection and set-union will be denoted by $\subseteq$ $\cap$ and U respectively.

An element a of a poset P with O is said to have the pseudocomplement a^* in P if there exists in P an element a^* such that (i) $(a] \subseteq (a^*] = (0]$ and (ii) if $(a] \cap (b] = (0]$ for b in P then $(b] \subseteq (a^*]$ *A* poset P is said to be pseudocomplemented if each of its element has a pseudocomplement. An element a of a poset P with 0 is said to be dense if $(a] \cap (b] = (0] \Rightarrow b = 0$ for $b \in P$. The set of all elements x of P such that $x \leqslant a$ for some fixed a in P forms an ideal of P. It is called the principal ideal generated by a and is denoted by $(a]$.

We need the following lemmas in sequel.

Lemma 1[5]—The set I of all ideals of a poset P with O is a complete lattice under set inclusion as ordering relation.

Lemma 2[5]—In a poset P a finite join $a_1 \vee a_2 \vee \ldots \vee a_n$ exists if and only if $(a] \vee (a_2] \vee \ldots \vee (a_n]$ is a principal ideal. Also whenever $a_1 \vee \ldots \vee a_n$ exists

$$(a_1 \vee \ldots \vee a_n] = (a_1] \vee (a_2] \vee \ldots \vee (a_n]$$

($\vee$ denotes the join in I_μ).

Lemma 3[5]—In a poset P with O the pseudocomplement a^* of an element a exists if and only if $(a]^* = \{x \in P/(x] \cap (a] = (0]\}$ is a principal ideal. Further whenever a^* exists, $(a]^* = (a^*]$.

Lemma 4[3]—Every distributive lattice with O (semilattice with O) is a O-distributive lattice (semilattice).

Lemma 5—Every pseudocomplemented semilattice is O-distributive. Throughout this paper the symbol P denotes a poset P with O.

2. *O*-DISTRIBUTIVE POSETS

We begin with

Definition 1—A poset P is called as a O-distributive poset if for $a, x_1, \ldots x_n \in P$ (n finite)

$$(a] \cap (x_i] = (\theta] \ \forall\ i,\ 1 \leqslant i \leqslant n \text{ imply } (a] \cap (x_1 \vee \ldots \vee x_n] = (0]$$

whenevvr $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in P.

Remark : It is clear that our definition concides with the definition of Pawar[3] in a semilattice.

Examples of O-distributive posets are given in the following figures.

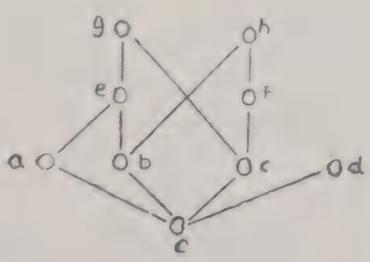

FIG. 1.

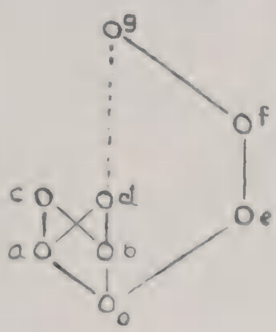

FIG. 2.

Examples—Note that every poset with O need not be O-distributive. The following is an example of a non-O-distributive poset.

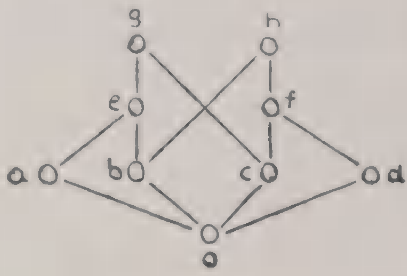

FIG. 3.

Example—In a poset P define

$$\{a\}^* = \{b \in P / (a] \cap (b] = (0]\}.$$

We characterize O-distributive poset as

Theorem 2—Poset P is O-distributive if and only if $\{a\}^*$ is an ideal for any a in P.

PROOF : Obviously in any poset $\{a\}^*$ is a semi-ideal. If for $x_1, \ldots, x_n$ (n finite) in $\{a\}^*$, $x_1 \vee \ldots \vee x_n$ exists then by O-distributivity $(a] \cap (x_1 \vee \ldots \vee x_n] = (0]$ proving that $x_1 \vee x_2 \vee \ldots \vee x_n \in \{a\}^*$ i.e. $\{a\}^*$ is an ideal. Conversely, if $\{a\}^*$ is an ideal then by the definition of an ideal, O-distributivity of P follows.

For any subset A of P If we denote $A^* = \{x \in P / (x] \cap (a] = (0]$ for all $a \in A\}$. Then obviously $A^* = \bigcap_{a \in A} \{a\}^*$. As arbitrary intersection of ideals is an ideal we get.

Corollary 3—P is O-distributive if and only if A^* is an ideal for any $A \subseteq P$.

By Lemma 3, it follows that P is pseudocomplemented if and only if $\{a\}^*$ is a principal ideal. Hence by Theorem 1 we get.

Corollary 4—Every pseudocomplemented poset is O-distributive.

The above corollary establishes that fact that O-distributive poset is a generalization of pseudocomplemented poset, as every O-distributive poset need not be pseudocomplemented. This is shown by a poset represented in the following figure.

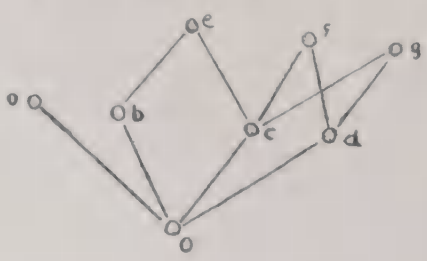

FIG. 4.

While generalizing the concept of disjunctivity to posets Venkatanarasimhan[6] defined disjunctive poset. Poset P is called disjunctive poset if $a \neq b$ in P implies the existence of c in P such that $(a] \cap (c] = (0]$ and $(b] \cap (c] \neq (0]$.

O-distributivity and disjunctivity are completely independent in a poset. This is cited by the following posets.

Example—Example of a poset which is O-distributive but not disjunctive.

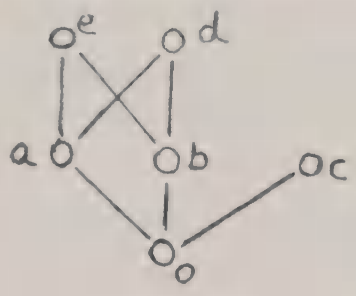

FIG. 5.

Example—Example of a poset which is disjunctive but not O-distributive.

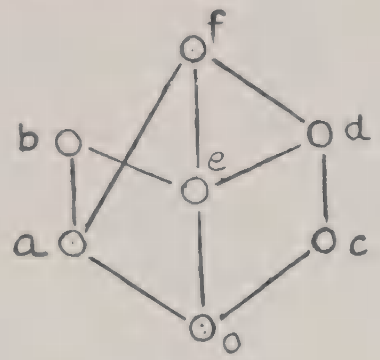

FIG. 6.

Example—Example of a poset which is neither O-distributive nor disjunctive.

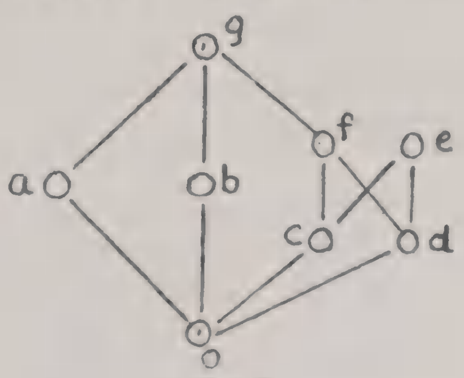

FIG. 7.

Every distributive lattice with O (semilattice with O) is O-distributive (see Lemma 4). Hence to keep up such a linking for posets, we define.

Definition 5—A poset Q is called as a distributive poset if $(a] \cap (b] \subseteq (c]$ $(a_1 b_1 c \in Q)$ implies the existence of x, y in Q $x \geqslant a, y \geqslant b$ such that $(x] \cap (y] = (c]$.

Theorem 6—Every distributive poset with 'o' is O-distributive.

PROOF : Let P be a distributive poset with O. Let $a, x_1, x_2, \ldots, x_n$ (n finite) be in P such that $(a] \cap (x_i] = (0]$ $\forall$ $1 \leqslant i \leqslant n$. Suppose that $x_1 \vee x_2 \ldots \vee x_n$ exists in P.

Now, $(x_1] \supseteq (x_2] \cap (a]$. Hence by distributivity there exist $y_2 \geqslant_1 x_2$ and $y_1 \geqslant a$ such that $(x_1] = (y_1] \cap (y_2]$. As $(y_1] \supseteq (y_1] \cap (y_2]$ we get

$$(y_1] \supseteq (x_1] \text{ i.e. } y_1 \geqslant x_1.$$

Further, $(x_1] \supseteq (x_r] \cap (a]$. Hence by distributivity there exist $y_r \geqslant x_r$ and $z_r > a$ such that

$$(x_1] = (y_r] \cap (z_r].$$

Thus we get, $y_1 \geqslant x_1, y_2 \geqslant x_2, \ldots\ldots, y_n \geqslant x_n$. Since the set of all ideals, I_μ, is a lattice (Lemma 1)

$$(y_1] \cap (y_2] \cap \ldots\ldots (y_n) \subseteq (x_1] \vee (x_2] \vee \ldots\ldots \vee (x_n]$$
$$= (x_1 \vee \ldots \vee x_n] \qquad \text{(see Lemma 2).}$$

Hence

$$(a] \cap \{(y_1] \cap (y_2] \cap \ldots (y_n]\} \supseteq (a] \cap \{(x_1] \vee (x_2] \vee \ldots \vee (x_n]\}$$

which in turn proves that

$$(a] \cap \{(x_1] \vee (x_2] \vee \ldots \vee (x_\mu]\} = (0].$$

Since $(a] \cap (y_2] = (0]$.

Hence $(a] \cap (x_1 \vee x_2 \vee \ldots \vee x_n] = (0]$ i.e. P is O-distributive.

Venkatanarasimhan[5] proved that for any poset I_μ is a complete lattice. (See Lemma 1). For I_μ to be pseudocomplemented we prove.

Theorem 7—P is O-distributive if and only if I_μ is pseudocomplemented.

PROOF : Let P be a O-distributive poset and $A \in I_\mu$. By Corollary 3 A^* is an ideal in P. We claim that A^* is the pseudocomplement of A in P. Clearly, $A \cap A^* = (0]$. If there exists B in I_μ such that $A \cap B = (0]$ then $B \subseteq A^*$. For $b \in B$ implies that $(a] \cap (b] = (0]$ for every a in A. This proves that I_μ is pseudocomplemented. Conversely, let I_μ be pseudocomplemented. For $a_1 x_1, x_2 \ldots x_n$ (n finite) in P suppose that $(a] \cap (x_i] = (0]$ for $i \leqslant i \leqslant n$. Assume that $x_1 \vee x_2 \ldots \vee x_n$ exists in P. By assumption, $(x_1] \subseteq (a]^* n$, and hence $(x_1] \vee (x_2] \vee \ldots \vee (x_n] \subseteq (a]^*$ But, by Lemma 2, $(x_1] \vee (x_2] \vee \ldots \vee (x_n] = (x_1 \vee x_2 \vee \ldots x_n]$. Hence

$(x_1 \vee x_2 \vee \dots \vee x_n] \subseteq (a]^*$ proving that $(x_1 \vee \dots \vee x_n] \cap (a] = (0]$. Therefore P is O-distributive.

Corollary 8—When P is a pseudocomplemented poset I_μ is pseudocomplemented.

As we know that every pseudocomplemented lattice is O-distributive (Lemma 5) one more generalization of O-distributivity is obtained. This is given in the following.

Theorem 9—P is O-distributive if and only if I_μ is O-distributive.

PROOF: Let I_μ be O-distributive. Let $x_1, x_2, \dots, x_n$ be in P such that $(x_i] \cap (a] = (0]$ for every i, $1 \leqslant i \leqslant n$. Assume that $x_1 \vee x_2 \vee \dots \vee x_n$ exists in P. As $(x_1] \cap (a] = (0], \dots (x_n] \cap (a] = (0]$ in I_μ and I_μ is O-distributive we get

$$(x_1] \vee (x_2] \vee \dots \vee (x_n] \cap (a] = (0]. \quad \text{But by Lemma 2}$$

$$(x_1] \vee (x_2] \vee \dots \vee (x_n] = (x_1 \vee x_2 \vee \dots \vee x_n]. \quad \text{Hence}$$

$$(x_1 \vee x_2 \vee \dots \vee x_n] \cap (a] = (0]$$

which in turn proves O-distributivity of P. Conversely, let P be O-distributive. By Theorem 7, I_μ is pseudocomplemented. But every pseudocomplemented lattice being O-distributive (see Lemma 5) I_μ is O-distributive.

A poset Q is said to satisfy the ascending chain condition if any increasing chain terminates in Q i e. if $x_1 \in P$, $i = 0, 1, 2, \dots$ and $x_0 \leqslant x_1 \leqslant \dots \leqslant x_n \dots$ then for some on we have $x_m = x_{m+1} = \dots\dots\dots$

Clearly, in a poset satisfying ascending chain condition every ideal is principal. Using this we prove.

Theorem 10—Every O-distributive poset satisfying ascending chain condition is pseudocomplemented.

PROOF: Let P be O-distributive poset satisfying ascending chain condition For any a in P, $(a]^*$ is an ideal in P, by Theorem 1. As P satisfies ascending chain condition, $(a]^*$ is a principal ideal. Hence P is pseudocomplemented. (See Lemma 3).

A sufficient condition for $(a]^* = (b]^*$ in a O-distributive poset for $a \neq b$ is stated in the following.

Theorem 11—If a and b are the elements of a O-distributive poset such that $(a] \cap (d] = (b] \cap (d]$ for some dense element $d E \in P$ then $(a]^* = (b]^*$.

PROOF: $(a]^{**} = (a]^{**} \cap P = (a]^{**} \cap (d]^{**}$ (d any dense element in P) $= \{(a] \cap (d]\}^{**} = \{(b] \cap (d]\}^{**} = (b]^{**} \cap (d]^{**} = (b]^{**} \cap P = (b]^{**}$. Hence $(a]^* = (b]^*$.

A property of the set of dense elements in a O-distributive poset is investigated in the following.

Theorem 12—In a O-distributive poset P if $\{0\} \neq A$ is the intersection of all nonzero ideals of P then $A^* = P - D$ where D is set of all dense elements of P.

PROOF : $A \neq \{0\}$ implies that $\{x\}^* \neq \{0\}$ for any x in P. i.e. $x \in A^* \Rightarrow x \in P - D$. Hence $A^* \subseteq P - D$. On the other hand, P being O-distributive, $\{d\}^*$ is a non-zero ideal of P for every $d \notin D$. But $A \subseteq \{d\}^*$ implies $A^* \supseteq d^{**}$. But $d \in \{d\}^{**}$ implies $d \in A^*$. Thus $P - D \subseteq A^*$. This proves that $A^* = P - D$.

3. ANNIHILATOR IDEALS

In this section we deal with annihilator ideals in a O-distributive poset.

Cornish[1] has defined annihilator ideal in a distributive lattice. On the same lines we define annihilator ideals in a O-distributive poset, as follows.

Definition—An ideal J of a O-distributive poset P is called an annihilator ideal if $J = J^{**}$ i.e. $J = S^*$ for some subset S of P.

The collection of all annihilator ideals in a O-distributive poset P is denoted by $A(P)$.

Theorem 13—For a O-distributive poset P, the set of all annihilator ideals $A(P)$ forms a Boolean algebra.

PROOF : For I and J in $A(P)$ define

$$I \wedge J = I \cap J \text{ and } I \vee J = (I^* \cap J^*)^*.$$

(i) As $I = I^{**}$ and $J = J^{**}$ we get $I \cap J$ is the g.l.b. of I and J. Further $(I \cap J)^{**} \supseteq (I \cap J)$ and $I \subseteq I^{**}$, $J \subseteq J^{**}$ implies $(I \cap J) \subseteq I^{**} \cap J^{**}$ proving that $I \cap J = (I \cap J)^{**}$ i.e. $I \cap J$ is in $A(P)$. Hence $I \cap J \in A(P)$ for I, J in $A(P)$.

(ii) Again $I, J \in A(P) \Rightarrow I \subseteq (I^* \cap J^*)^*$ and $J \subseteq (I^* \cap J^*)^*$. If $I \subseteq K$ and $J \subseteq K$ for some $K \in A(P)$ then $I^* \supseteq K^*$, $J^* \supseteq K^*$ will imply $I^* \cap J^* \supseteq K^*$ i.e. $(I^* \cap J^*)^* \subseteq K^{**} = K$. But then this shows that $(I^* \cap J^*)^*$ is the l.u.b. of I and J in $A(P)$. Hence $I \vee J \in A(P)$.

From (i) and (ii) we get $\langle A(P); \wedge, \vee \rangle$ is a lattice.

Since $(0] = P^*$ and $P = (0]^*$, $(0]$ and P are the elements of $A(P)$. Further $(0]$ and P are the least and the greatest elements of $A(P)$.

Thus $A(P)$ is a bounded lattice.

Next we show that $A(P)$ is complemented. Let $I \in A(P)$. Then obviously $I^* \in A(P)$. Further $I \vee I^* = (I^* \cap I^{**})^* = (I^* \cap I)^* = (0]^* = P$ and $I \cap I^* = (0]$ show that I^* is the complement of I in $A(P)$.

It only remains to show $A(P)$ is distributive that, for $I, J, K \in A(P)$ we have to show that

$$I \vee (J \wedge K) = (I \vee J) \wedge (I \vee K)$$

But $I \vee (J \wedge K) \leqslant (I \vee J) \wedge (I \vee K)$ is true always.

Hence we have to only prove that

$$(I^* \cap J^*)^* \cap (I^* \cap K^*)^* \subseteq [I^* \cap (J \cap K)^*]^*.$$

To prove this we need to prove the following set inclusion

$$(I^* \cap J^*)^* \cap K \subseteq [I^* \cap (J \cap K)^*]^*.$$

Let $I, J, K \in A(P)$. Now $I \cap K \subseteq I \subseteq [I^* \cap (I \cap K)^*]^*$.

Similarly $J \cap K \subseteq [I^* \cap (J \cap K)^*]^*$.

Now $I \cap K \subseteq [I^* \cap (J \cap K)^*]^* \Rightarrow I \cap K \cap [I^* \cap (J \cap K)^*]^{**} = (0]$

that is $I \cap K \cap [I^* \cap (J \cap K)^*] = (0]$.

Silimarly $J \cap K \cap [I^* \cap (J \cap K)^*] = (0]$

that is $J \cap [K \cap I^* \cap (J \cap K)^*] = (0]$.

But this imply

$$[K \cap I^* \cap (J \cap K)^*] \subseteq J^*. \quad \text{Similarly}$$

$$[K \cap I^* \cap (J \cap K)^* \subseteq I^*$$

$\Rightarrow [K \cap I^* \cap (J \cap K)^*] \subseteq I^* \cap J^*$

$\Rightarrow [K \cap I^* \cap (J \cap K)^*] \cap (I^* \cap J^*)^* = (0]$

that is $I^* \cap (J \cap K)^* \cap [K \cap (I^* \cap J^*)^*] = (0]$

$\Rightarrow K \cap (I^* \cap J^*)^* \subseteq [I^* \cap (J \cap K)^*]^*$

i.e. $(I^* \cap J^*)^* \cap K \subseteq [I^* \cap (J \cap K)^*]^*$

i.e. $(I \vee J) \wedge K \leqslant I \vee (J \wedge K)$ providing that $A(P)$ is distributive.

Thus $A(P)$ is a complemented, distributive, lattice and hence a Boolean algebra.

References

1. W. H. Cornish, *J. Aust. Math. Soc.*, **15** (1973), 70-77.
2. O. Frink, *Am. Math. Monthly* **61** (1954), 223–34; *MR* **15**, 848.
3. Y. S. Pawar, Doctoral thesis.Shivaji University, Kolhapur, 1978.
4. J. C. Varlet, *Bull. Soc. Roy. Sci. Liege* **41** (1972).
5. P. V. Venkatnarasimhan, *Proc. Am. Math Soc.* **23** (1971), 9-17; *MR* **42**, 7368.
6. P. V. Venkatnarasimhan, *Colloq. Math.* **30** (1974), 203 13; *MR* **50**, 128–38.

Indian J. pure appl. Math., **20** (8) : 812–819, August 1989

FUNCTIONAL LIMITS

G. DAS AND S. NANDA*

Department of Mathematics, Utkal University, Bhubaneswar 751004

(*Received* 6 *May* 1987; *after revision* 19 *September* 1988)

The concepts of Banach limits and almost convergence of functions have been discussed respectively by Raimi[8] and Tien-Kung Ho[4]. The purpose of this paper is to introduce the continuous analogue of (Z, p) summable sequences. Several properties of such functions including their relationship with almost convergent functions are discussed.

INTRODUCTION

Let E be a Banach space of measurable essentially bounded real-valued functions defined on the real line R with the norm $\|f\| = \text{ess. sup} \{ |f(x)| : x \in R\}$ and let L be the class of functions Lebesgue integrable in every finite interval. That E and L are actually made up of equivalent classes of such functions will be ignored in the sequel. For $c \in R$ and $f \in E$ we define f_c by

$$f_c(x) = f(x + c).$$

We assume that E contains all bounded uniformly continuous functions and the functions f_c for each $f \in E$ and $c \in R$. Let e denote the constant function defined by $e(x) = 1$ for all $x \in R$.

Let E^* denote the conjugate space of E. An element $\Phi \in E^*$ is said to be a Banach limit (see Raimi[8]) if

$$\|\Phi\| = 1$$

$$\Phi(e) = 1$$

and

$$\Phi(f_c) = \Phi(f) \text{ for all } f \in E \text{ and } c \in R.$$

This is the continuous analogue of Banach limits for bounded sequences (see Banach[1] and Lorentz[7]).

It has been shown by Raimi[8] that if f is a bounded uniformly continuous function, then all its Banach limits coincide and equal to $\lim\limits_{x \to \infty} f(x)$† .

†In the sequal $\alpha, x \to \infty$ in the limits.

*On leave of absence from Indian Institute of Technology, Khargpur 721302.

For each $\alpha \geqslant 0$ and $x \in R$ we define the operator T_α by

$$T_\alpha f(x) = \begin{cases} f(x) & (\alpha = 0) \\ \dfrac{1}{\alpha} \displaystyle\int_x^{x+\alpha} f(t)\, dt & (\alpha > 0). \end{cases}$$

Let F denote the set of all almost convergent functions, that is

$$F = \{ f \in E : \lim_{\alpha \to \infty} T_\alpha f(x) \text{ exists uniformly in } x \}.\dagger$$

This has been discussed by Raimi[8] and Hö[4]. Note that F is the continuous analogue of almost convergent sequences [see Lorentz[3]]. It has been demonstrated by Raimi[8] that if $f \in F$, then all its Banach limits coincide with $\lim_\alpha T_\alpha f(x)$.

We now introduce the set

$$S_\alpha = \{ f \in L : \lim_{x \to x} T_\alpha f(x) \text{ exists for fixed } \alpha \geqslant 0 \}.$$

It is obvious that

$$S_0 = \{ f \in L : \lim_x f(x) \text{ exists} \}.$$

It should be noted here that $T_\alpha f(x)$ is a special case of the functional Nörlund transform of $f(t + \alpha)$ generated by the function

$$p(t) = \begin{cases} 1 & (0 \leqslant t \leqslant \alpha) \\ 0 & (t > \alpha). \end{cases} \qquad \text{...(1)}$$

Although the general functional Nörlund methods have been discussed by Knopp and Vanderburg[5] and Knopp[6], but still there is scope for considering the special and interesting method defined by (1) which has so far not been studied. The series analogue of this special method has been discussed by Silverman and Szasz[9], Hill and Sledd[3] and Das[2]. If $f \in S_\alpha$ is such that $T_\alpha f(x) \to s$ as $x \to \infty$ for fixed α, then we sometimes write this as $f(x) \to s\ (S_\alpha)$. It should be remarked here that f could be in S_α even if it is not essentially bounded and is merely integrable in any finite range, whereas it should be essentially bounded in order to be in F.

We define the set

$$| S_\alpha | = \{ f \in L : T_\alpha f(x) \in BV_x \text{ for fixed } \alpha \}.$$

Then

$$| S_0 | = \{ f \in L : f(x) \in BV \}.$$

It is obvious that $| S_\alpha | \subset S_\alpha$ for every $\alpha \geqslant 0$.

If we write

†In the sequal α, $x \to \infty$ in the limits.

$$C_1 = \left\{ f \in L : \lim_{\alpha} T_\alpha f(0) = \lim_{\alpha} \frac{1}{\alpha} \int_0^\alpha f(t)\, dt \text{ exists} \right\}$$

then it is obvious that $F \subset C_1 \cap E$.

The purpose of the present paper is to study the properties of the sets S_α and $|S_\alpha|$ only.

The Results

Theorem 1—The translation of the argument of f has no influence on its generalised limits, that is, for a constant $c > 0$,

$$f(x) \to s\ (S_\alpha) \Leftrightarrow f(x+c) \to s\ (S_\alpha),\ \alpha \geqslant 0.$$

Proof : Let

$$\bar{T}_\alpha f(x) = \frac{1}{\alpha} \int_x^{x+\alpha} f(t+c)\, dt.$$

Hence

$$\bar{T}_\alpha f(x+c) = T_\alpha f(x)$$

and this proves the theorem.

Theorem 2—$S_0 \subset S_\alpha$ for every $\alpha > 0$.

$|S_0| \subset |S_\alpha|$ for every $\alpha > 0$.

Proof : Since, for given $\epsilon > 0$ and for $x \geqslant x_0$, $|f(x) - s| < \epsilon$ implies

$$\left| \frac{1}{\alpha} \int_0^\alpha f(x+t)\, dt - s \right| \leqslant \frac{1}{\alpha} \int_0^\alpha \left| f(x+t) - s \right| dt < \epsilon$$

it follows that $S_0 \subset S_\alpha$. In fact

$$S_0 = \bigcap_{\alpha \geqslant 0} S_\alpha .$$

It is easy to verify that

$$T_\alpha f(x) = T_\alpha f(0) + \frac{1}{\alpha} \int_0^x \{ f(t+\alpha) - f(t) \}\, dt. \qquad ...(2)$$

Thus it follows from (2) that $f(x) \to s\ (S_\alpha)$ if and only if

$$\int_0^x \{f(t+\alpha) - f(t)\}\, dt \to \alpha\,(s - T_\alpha f(0)) \text{ as } x \to \infty. \qquad \ldots(3)$$

It follows from (3) that the definition of $|S_\alpha|$ is equivalent to

$$\int_0^\infty |f(t+\alpha) - f(t)|\, dt < \infty \;(\alpha > 0).$$

If we write for $g \in L$, $f(t) = \int_0^t g(u)\, du$, then

$$\begin{aligned}\int_0^\infty |f(t+\alpha) - f(t)|\, dt &= \int_0^\infty dt \left| \int_0^\alpha g(u+t)\, du \right| \\ &\leqslant \int_0^\alpha du \int_0^\infty |g(u+t)|\, dt \\ &\leqslant \int_0^\alpha du \int_0^\infty |g(t)|\, dt \\ &= \int_0^\alpha du \int_0^\infty |df(t)| \\ &\leqslant \alpha V(f) < \infty.\end{aligned}$$

This shows that $|S_0| \subset |S_\alpha|$ for every $\alpha > 0$. In fact

$$|S_0| = \bigcap_{\alpha \geqslant 0} |S_\alpha|\,.$$

Theorem 3—If $T_\alpha f(x) \to s$ and $T_\beta f(x) \to s'$ as $x \to \infty$, then $s = s'$.

PROUF : This follows from Theorem 2 and from the fact that

$$T_\alpha (T_\beta f(x)) = T_\beta (T_\alpha f(x)).$$

Because of Theorem 3 the statment $S_\beta \subset S_\alpha$ means that if $f(x) \to s\ (S_\beta)$, then $f(x) \to s\ (S_\alpha)$.

At this stage we make a conjecture :

Conjecture 4—$S_\alpha \cap E \subset F$.

We now prove :

Theorem 5—$S_\alpha \subset C_1\ (\alpha \geqslant 0)$.

PROOF : If $\alpha = 0$, the theorem is obvious. Suppose now that $0 < \alpha \leqslant 1$. We write $N = [x]$. Then

$$\frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x}\left(\int_0^{N\alpha} + \int_{N\alpha}^x\right) f(t)\,dt \qquad \ldots(4)$$

Now

$$\int_0^{N\alpha} f(t)\,dt = \sum_{r=0}^{n-1}\int_0^{\alpha} f(t + r\alpha)\,dt.$$

Without loss of generality we take $s = 0$. Hence by hypothesis

$$\theta_r = \frac{1}{\alpha}\int_0^{\alpha} f(t + r\alpha)\,dt \to 0 \text{ as } r \to \infty$$

and so

$$\frac{1}{N}\sum_{r=0}^{N-1} \theta_r \to 0 \text{ as } N \to \infty.$$

Since $\frac{N}{x} \to 1$ as $x \to \infty$ we get

$$\frac{1}{x}\int_0^{N\alpha} f(t)\,dt \to 0 \text{ as } x \to \infty. \qquad \ldots(5)$$

Now we have to show that

$$\frac{1}{x}\int_{N\alpha}^{x} f(t)\,dt \to 0 \text{ as } x \to \infty. \qquad \ldots(6)$$

If $\alpha = 1$, then (6) obviously holds. If $0 < \alpha < 1$, we have

$$\frac{1}{x}\int_{N\alpha}^{x} f(t)\,dt = \frac{1}{x}\sum_{r=1}^{m}\int_0^{\alpha} f(t + [x]\alpha + (r-1)\alpha)\,dt$$

$$+ \frac{1}{x}\int_{[x]\alpha+m\alpha}^{x} f(t)\,dt \qquad \ldots(7)$$

where m is an integer such that

$$[x]\,\alpha + m\,\alpha \leqslant x < [x]\,\alpha + (m+1)\,\alpha.$$

Clearly

$$\frac{1}{x}\int_{[x]\alpha+m\alpha}^{x} f(t)\,dt = o\left(\frac{1}{x}\right) = o(1), \text{ as } x \to \infty. \qquad \ldots(8)$$

Since $T_\alpha\, f(x) \to 0$ as $x \to \infty$, choose $\epsilon > 0$ and m_0 such that, for $r > m_0$

$$\int_0^\alpha f(t + [x]\,\alpha + (r-1)\,\alpha)\,dt < \epsilon.$$

Hence

$$\frac{1}{x}\sum_{r=1}^{m}\int_0^\alpha f(t + [x]\,\alpha + (r-1)\,\alpha)\,dt$$

$$= \frac{1}{x}\left(\sum_{r=1}^{m_0} + \sum_{r=m_0+1}^{m}\right)\int_0^\alpha f(t + [x]\,\alpha + (r-1)\,\alpha)\,dt$$

$$= o(1/x) + o(m/x) = o(1).$$

Now the theorem follows from (4) – (8).

When $\alpha > 1$ the theorem can be proved similarly.

On the contrary, we have

Theorem 6—For $\alpha > 0$,

$$f \in C_1 \Rightarrow T_\alpha f(x) = o(x).$$

PROOF : It follows from (2) that

$$\alpha\,\frac{d}{dx}\,T_\alpha\, f(x) = f(x+\alpha) - f(x)$$

almost everywhere. Hence

$$\int_0^x f(t)\,dt = \int_0^x \left\{ f(t+\alpha) - \alpha\,\frac{d}{dt}\,T_\alpha\, f(t)\right\} dt$$

(equation continued on p. 818)

$$= \int_0^x f(t+\alpha)\, dt - \alpha \int_0^x \frac{d}{dt} T_\alpha f(t)\, dt$$

$$= \int_0^x f(t+\alpha)\, dt - \alpha \{T_\alpha f(x) - T_\alpha f(0)\}. \qquad \text{...(9)}$$

It follows from (9) that when $f \in C_1$, then

$$T_\alpha f(x) - T_\alpha f(0) = o(x)$$

and this completes the proof of the theorem.

The following theorems give inclusion relations for S_α.

Theorem 7—If α is an integral multiple of β, then $S_\beta \subset S_\alpha$.

PROOF : Suppose that $\alpha = m\beta$ where m is an integer. We exclude the cases when $\beta = 0$ or $\alpha = \beta$ which are trivial. Now suppose that $f \in S_\beta$ such that $f(x) \to s(S_\beta)$. We have

$$T_\alpha f(x) = \frac{1}{\alpha} \int_x^{x+m\beta} f(t)\, dt$$

$$= \frac{1}{\alpha} \sum_{r=1}^{m} \int_{x+(r-1)\beta}^{x+r\beta} f(t)\, dt$$

$$= \frac{1}{\alpha} \sum_{r=1}^{m} \int_x^{x+\beta} f(t+(r-1)\beta)\, dt$$

$$\to \frac{\beta}{\alpha} \sum_{r=1}^{m} s = s$$

(by Theorem 1 and the hypothesis). Hence $f \in S_\alpha$.

Theorem 8—(i) $S_\alpha \cap S_\beta \subset S_{\alpha+\beta}$

(ii) $S_\alpha \cap S_\beta \subset S_{\alpha-\beta}\ (\alpha > \beta)$.

PROOF : We have from Theorems 1 and 3 that

$$(\alpha+\beta)\ T_{\alpha+\beta} f(x) = \left(\int_x^{x+\alpha+\beta} + \int_x^{x+\alpha} + \int_{x+\alpha}^{x+\alpha+\beta} \right) f(t)\, dt$$

$$\to \alpha s + \beta s$$

and this proves (i). It is easily verified that for $\alpha > \beta$

$$\frac{\alpha T_\alpha f(x) - \beta T_\beta f(x)}{\alpha - \beta} = \frac{1}{\alpha - \beta} \int_x^{x+\alpha+\beta} f(t+\beta)\, dt. \qquad \ldots(10)$$

Now suppose that $f(x) \to s\ (S_\beta)$. Hence by Theorem 3 the left hand side of the expression in (10) converges to

$$\frac{\alpha s - \beta s}{\alpha - \beta} = s.$$

Now by Theorem 1 it follows that $f(x) \to s\ (S_{\alpha-\beta})$ and this completes the proof.

Theorem 9—Let $\alpha, \beta > 0$ be such that $\alpha = p\delta$, $\beta = q\delta$, and p, q are positive integers with $(p, q) = 1$. Then $S_\alpha \cap S_\beta = S_\delta$.

PROOF : Suppose that $f \in S_\delta$. Then by Theorem 7, $f \in S_{p\delta} = S_\alpha$ and $f \in S_{q\delta} = S_\beta$. Hence $f \in S_\alpha \cap S_\beta$. We have now to show that $S_\alpha \cap S_\beta \subset S_\delta$. Choose positive integers h and k such that $ph - qk = 1$. Hence $\delta ph - \delta qk = \alpha h - \beta k = \delta$. Now suppose that $f \in S_\alpha \cap S_\beta$. Hence by Theorem 7.

$$f \in S_\alpha \subset S_{\alpha h} \text{ and } f \in S_\beta \subset S_{\beta k}$$

and by Theorem 8,

$$f \in S_{\alpha h - \beta k} = S_\delta.$$

This completes the proof.

Remark : It may be observed that the results of Theorems 7, 8 and 9 remain true if the set S_α is replaced by $| S_\alpha |$.

ACKNOWLEDGEMENT

The authors wish to thank the referee for several valuable suggestions which improved the presentation of the paper.

REFERENCES

1. S. Banach, *Theorie des Operations Linearies*, Warszawa, 1932.
2. G. Das, *J. London Math. Soc.* (2) 6 (1973), 337–47.
3. J. D. Hill and W. T. Sledd, *Canad. J. Math* **16** (1964), 741–54.
4. Tien-Kung Ho, *Tamkang J. Math.* 6 (1975), 285-89.
5. K. Knopp and B. Vanderburg, *Rend. Circolo Math. Palermo Series II* Tomo IV (1955).
6. K. Knopp, *Math. Z.* 63 (1966), 39-52.
7. G. G. Lorentz, *Acta Math.* 80 (1948), 167–90.
8. Ralph A. Raimi, *Proc. Am. Math. Soc.* **8** (1957), 1029-36.
9. L. L. Silverman and O. Szasz, *Annal. Math.* **45** (1949), 247–57.

Indian J. pure appl. Math., **20** (8) : 820–828, August 1989

EXTREME POINTS OF SOME FAMILIES OF ANALYTIC FUNCTIONS RELATED TO UNIVALENT FUNCTIONS

A. K. MISHRA

School of Mathematical Sciences, Sambalpur University, Jyoti Vihar 768019

AND

P. SAHU

Department of Statistics, G. M. College, Sambalpur

(*Received* 2 *February* 1988; *after revision* 3 *November* 1988)

The concept of a^*-families has been introduced by Kapoor and Mishra[1] and Mishra[2]. In this paper some new results for functions in a^*-families with negative Taylor coefficients have been obtained. These results include coefficient characterization theorem, coefficient estimates, identification of extreme points and characterization for connectedness of a^*-families.

1. INTRODUCTION

Let $\mathscr{A}$ denote the class of functions f analytic in the unit disc $E = \{z: |z| < 1\}$, safisfying $f(0) = 0$ and $f'(0) \neq 0$. For $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ the Hadamard product $f * g$ of f and g is defined by $(f * g)(z) = \sum_{k-1}^{\infty} a_k b_k z^k$. Note that $f * g$ is also in $\mathscr{A}$.

Definition 1—Let s and g in $\mathscr{A}$ be given by

$$\left.\begin{aligned} s(z) &= \sum_{k=1}^{\infty} c_k z^k,\ c_1 > 0,\ c_k \geq 0 \\ g(z) &= \sum_{k=1}^{\infty} d_k z^k,\ d_1 > 0,\ d_k \geq 0 \end{aligned}\right\} \quad k = 2, 3 \ldots \qquad \ldots(1) \ldots(2)$$

with

$$(c_k/c_1) - (d_k/d_1) > 0 \quad k = 1, 2, 3 \ldots \qquad \ldots(3)$$

We denote by $F(s, g, \alpha)$, $0 \leqslant \alpha \leqslant (c_1/d_1)$, the class of functions f in $\mathscr{A}$ satisfying

$$(g * f)(z) \neq 0, \ 0 < |z| < 1 \qquad \ldots(4)$$

and

$$\operatorname{Re}\left[\frac{(s * f)(z)}{(g * f)(z)}\right] > \alpha,\ z \in E. \qquad \text{...(5)}$$

Definition 2—Let q given by

$$q(z) = 1 + \sum_{k=2}^{\infty} e_k z^{k-1} \qquad \text{...(6)}$$

be analytic in E We denote by $F(s, g, \alpha, q, z_0)$, z_0 real, $0 < |z_0| < 1$, the class of functions f in $F(s, g, \alpha)$ satisfying

$$\left(\frac{f}{z} * q\right)(z_0) = 1. \qquad \text{...(7)}$$

If B is a subset of $-1 < z < 1$, $z \neq 0$, then $F(s, g, \alpha, q, B)$ stands for $\bigcup_{z_i \in B}$ $F(s, g, \alpha, q, z_i)$. Further, we denote by $F[s, g, \alpha]$ (respectively $F[s, g, \alpha, q, B]$) the class of functions f in $F(s, g, \alpha)$ (respectively in $F(s, g, \alpha, q, B)$) given by the series

$$f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geqslant 0,\ k = 1, 2, 3 \ldots \qquad \text{...(8)}$$

The class $F(s, g, \alpha)$ is introduced and studied in Kapoor and Mishra[2] and Mishra[3] where it is called an a^*-family. Suitable choices of the touple give us many familiar families related to univalent functions. For example :

(*a*) $F\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha\right)$ is the class of starlike function of order α.

(b) $F\left(\frac{z(1+z)}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha\right)$ is the class of convex functions of order α.

(c) $F\left(\frac{z}{(1-z)^2}, \frac{z}{1-z^2}, \alpha\right)$ is the class of functions starlike with respect to symmetric points and of order α (Sakaguchi[5]).

(d) $F\left(\frac{z}{(1-z)^2}, z, \alpha\right)$ is an extensively studied subclass of close-to-convex functions.

(e) $F\left(\frac{z}{(1-z)^{n+2}}, \frac{z}{(1-z)^{n+1}}, \frac{1}{2}\right)$ $n = 0, 1, 2, \ldots$, is a class of functions studied in Goel and Sohi[1]

(f) $F\left(\frac{z}{(1-z)^{2(1-\alpha)+1}}, \frac{z}{(1-z)^{2(1-\alpha)}}, \frac{1}{2}\right)$, $0 \leqslant \alpha < 1$, is the class of prestarlike functions of order α Ruscheweyh[4].

(g) $F\left(\frac{z}{1-z}, z, \alpha\right)$ is the class of functions for which $\mathrm{Re}\,\{f(z)/z\} > \alpha$.

In Definition, 2, if we take $q(z) = \frac{1}{1-z}\left(\text{respectively } q(z) = \frac{1}{(1-z)^2}\right)$ then $F(s, g, \alpha, q, z_0)$ is the subclass of functions f in $F(s, g, \alpha)$ satisfying $f(z_0) = z_0$ (respectively $f'(z_0) = 1$). Finally, let G be a family of functions analytic in E. A function f in G is said to be an extreme point of G if $f \neq tg_1 + (1-t)g_2$ for $0 < t < 1$ and any pair of distinct functions g_1 and g_2 in $\mathscr{H}$.

Silverman[6,7] first studied the particular a^*-families

$$F\left[\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha\right], F\left[\frac{z(1+z)}{(1+z)^3}, \frac{z}{(1-z)^2}, \alpha\right]$$

$$F\left[\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \frac{1}{1-z}, z_0\right], F\left[\frac{z}{(1-z)^2} \frac{z}{1-z}, \alpha, \frac{1}{(1-z)^2}, z_0\right]$$

$$F\left[\frac{z(1+z)}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \frac{1}{1-z}, z_0\right] \text{and}$$

$$F\left[\frac{z(1+z)}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \frac{1}{(1-z)^2}, z_0\right]$$

and determined their extreme points. Subsequently Silverman and Silvia[8] have also investigated the a^*-family

$$F\left[\frac{z}{(1-z)^{2(1-\alpha)+1}} \frac{z}{(1-z)^{2(1-\alpha)}}, \tfrac{1}{2}\right]$$

and have determined its extreme points. In recent years there has been extensive study in the determination of extreme points of families of functions with negative coefficients. Kapoor and one of the present authors[2,3] have determined the extreme points of any a^*-family $F[s, g, \alpha]$. In the present paper we determine the extreme points of $F[s, g, \alpha, q, z_0]$ for arbitrary choice of the touple (s, g, α, q, z_0). We also show that if B is a subset of $(0, 1)$ then $F[s, g, \alpha, q, B]$ is a convex family if and only if B is connected. We determine the extreme points of $F[s, g, \alpha, q, B]$ for connected B.

2. Extreme Points of $F[s, g, \alpha, q, z_0]$

We need the following theorem from Kapoor and Mishra[2] (also see Mishra[3]).

Theorem A—Let $\{c_k\}$ and $\{d_k\}$ be sequences of non negative numbers with $c_1 > 0$, $d_1 > 0$, $(c_k/c_1) - (d_k/d_1) > 0$ and let $0 \leqslant \alpha \leqslant (c_1/d_1)$. Then a function f given by (8) is in $F[s, g, \alpha]$ where s and g are as in Definition 1 if and only if

$$\sum_{k=2}^{\infty} (c_k - \alpha d_k)\, a_k \leqslant \alpha_1 (c_1 - \alpha d_1). \quad ...(9)$$

The next theorem, a direct consequence of Theorem A, is useful for further investigations.

Theorem 1—Let s, g and α be as in Theorem A, z_0 be a real number such that $0 < |z_0| < 1$ and let $q(z)$ be as in Definition 2. Then the function $f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geqslant 0$, $k = 1, 2, 3, \ldots$ is in $F[s, g, \alpha, q, z_0]$ if and only if

$$\sum_{k=2}^{\infty} \left(\frac{c_k - \alpha d_k}{c_1 - \alpha d_1} - c_k z_0^{k-1} \right) a_k \leqslant 1. \quad ...(10)$$

PROOF : Observe that $f(z)$ satisfies the condition (7) if and only if $a_1 = 1 + \sum_{k=2}^{\infty} a_k e_k z_0^{k-1}$. Now, substituting this value of a_1 in (9) the result follows.

See that each term in the summation in (10) in nonnegative. This can be seen using (3) and the condition $0 \leqslant \alpha \leqslant (c_1/d_1)$.

Corollary 1—Let $f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geqslant 0$, be in $F[s, g, \alpha, q, z_0]$ where s, g, α, q and z_0 are as in Theorem 1. Then

$$a_k \leqslant \frac{c_1 - \alpha d_1}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_0^{k-1}}, \; k = 2, 3, \ldots \quad ...(11)$$

with equality for

$$f_k(z) = \frac{(c_k - \alpha d_k) z - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_0^{k-1}}.$$

$F[s, g, \alpha, q, z_0]$ is a convex family. For, if f and g are in $F[s, g, \alpha, q, z_0]$ and $0 < \lambda < 1$, then the function $\lambda f + (1 - \lambda) g$ satisfies the coefficient inequality (9) and the condition (7).

Theorem 2—Let the touple (s, g, α, q, z_0) be as in Definition 2.

Set

$$f_1(z) = z$$

and

$$f_k(z) = \frac{(c_k - \alpha d_k)\, z - (c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, z_0^{k-1}}, \quad k = 2, 3, \ldots \tag{12}$$

Then, a function f is in $F[s, g, \alpha, q, z_0]$ if, and only if it can be expressed in the form $\sum_{k=1}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof : Suppose $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Then,

$$f(z) = \left[\lambda_1 + \sum_{k=2}^{\infty} \frac{\lambda_k (c_k - \alpha d_k)}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}\right] z$$

$$- \sum_{k=2}^{\infty} \lambda_k \frac{(c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}.$$

We note that $\left(\frac{f}{z} * q\right)(z_0) = \sum_{k=1}^{\infty} \lambda_k \left(\frac{f_k}{z} * q\right)(z_0) = \sum_{k=1}^{\infty} \lambda_k = 1.$

Also,

$$\sum_{k=2}^{\infty} \frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}{(c_1 - \alpha d_1)} \lambda_k$$

$$\frac{(c_1 - \alpha d_1)}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}$$

$= \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leqslant 1.$ Hence f is in $F[s, g, \alpha, q, z_0]$.

Conversely, suppose that $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ is in $F[s, g, \alpha, q, z_0]$. For $k = 2, 3, \ldots$, write

$$\lambda_k = \frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}\, a_k}{(c_1 - \alpha d_1)}$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.$$

Observe that by Theorem 1, $\lambda_k \geqslant 0$, $k = 2, 3, \ldots$ and $\sum_{k=2}^{\infty} \lambda_k \leqslant 1$. Now $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$.

From Theorem 2 it follows that the extreme points of $F[s, g, \alpha, q, z_0]$ are precisely, the set of functions $\{f_k\}_{k=1}^{\infty}$ where f_k are defined as in (12).

3. The Family $F[s, g, \alpha, q, B]$

In this section we determine the extreme points of $F[s, g, \alpha, q, B]$ for a connected B. We have the following

Lemma 1—If $f \in F[s, g, \alpha, q, z_0] \cap F[s, g, \alpha, q, z_1]$ where s, g, α, and q are as in Theorem 1 and z_0 and z_1 are distinct positive numbers then $f(z) = z$.

Proof : Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $a_k > 0$. Then we must have

$$a_1 = 1 + \sum_{k=2}^{\infty} a_k e_k z_0^{k-1} = 1 + \sum_{k=2}^{\infty} a_k e_k z_1^{k-1}.$$

But this means that $a_k \equiv 0$ for $k \geqslant 0$.

Theorem 3—If B is contained in the interval (0, 1) then, $F[s, g, \alpha, q, B]$ is a convex family if and only if B is connected.

Proof : Let B be connected and let z_0 and z_1 be in B with $z_0 \leqslant z_1$. If $f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k$ is in $F[s, g, \alpha, q, z_0]$ and $g(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k$ is in $F[s, g, \alpha, q, z_1]$ and $0 \leqslant \lambda \leqslant 1$, then we shall show that there exists a z_2 in B $(z_0 < z_2 < z_1)$ such that $h(z) = \lambda f(z) + (1 - \lambda) g(z)$ is in $F[s, g, \alpha, q, z_2]$. Note that the coefficients of $h(z)$ satisfy (9). Now, write,

$$t(z) = \left(\frac{h}{z} * q\right)(z_0) = \lambda a_1 + (1 - \lambda) b_1 - \lambda \sum_{k=2}^{\infty} a_k e_k z^{k-1}$$

(*equation continued on p.* 826)

$$-(1-\lambda)\sum_{k=2}^{\infty} b_k e_k z^{k-1} = 1+\lambda\sum_{k=2}^{\infty} a_k e_k \left(z_0^{k-1} - z^{k-1}\right)$$

$$+(1-\lambda)\sum_{k=2}^{\infty} b_k e_k \left(z_1^{k-1} - z^{k-1}\right).$$

Observe that $t(z)$ is real for real z and $t(z_0) > 1$ and $t(z_1) < 1$. Hence there exists a z_2, $z_0 < z_2 < z_1$ such that $t(z_2) = 1$ and it follows that $h(z)$ is in $F[s, g, \alpha, q, z_2]$.

Conversely, if B is not connected we can choose z_0, z_1 in B and z_2 hot in B with $z_0 < z_2 < z_1$. Assume that f and g are not both identity functions. We write $t(\lambda) = t(z, \lambda) = ((\lambda f(z) + (1-\lambda) g(z))/z) * q(z)$ as above where $0 \leqslant \lambda \leqslant 1$. Now with z_2 fixed, $t(z_2, 0) > 1$ and $t(z_2, 1) < 1$. Hence, there exists a λ_0, $0 < \lambda_0 < 1$ such that $t(z_2, \lambda_0) = 1$ and it follows that $h(z) = \lambda_0 f(z) + (1-\lambda_0 g_0)$ is in $F[s, g, \alpha, q, z_2]$. But then, by Lemma 1, $F[s,g,\alpha,q,B]$ is not convex.

Theorem 4—Let $[z_0,z_1] \subset (0, 1)$. Then the extreme points of $F[s,g,\alpha, q, z_0]$ are z

$$f_k(z) = \frac{(c_k - \alpha d_k) z - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_0^{k-1}}$$

and

$$k = 2, 3, \ldots.$$

$$g_k(z) = \frac{(e_k - \alpha d_k) z - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_1^{k-1}}$$

Proof : Let f be an extreme point of $F[s, g, \alpha, q, B]$. Then, f must be an extreme point of $F[s, g, \alpha, q, z_2]$ for some z_2, $z_0 \leqslant z_2 \leqslant z_1$. We first show that either $z_2 = z_0$ or $z_2 = z_1$.

This we do by showing that if $z_0 < z_2 < z_1$ then

$$h_k(z) = \frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1) z_k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_2^{k-1}}$$

can be written as a convex combination of f_k and g_k. Write $h_k(z, \lambda) = \lambda f_k(z) + (1-\lambda) g_k(z)$. For z real and positive we have $h_k(z, 0) > h_k > h_k(z, 1)$. Hence, there exists a λ_0, $0 < \lambda_0 < 1$, for which $h_k(z, \lambda_0) = h_k(z)$. Infact, for λ_0 such that

$$\frac{1}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_2^{k-1}} = \frac{\lambda_0}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_0^{k-1}}$$

$$+ \frac{(1-\lambda_0)}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_1^{k-1}}$$

the coefficients of $h(z, \lambda_0)$ and $h_k(z)$ agree for all z. That is $h_k(z, \lambda_0) = h_k(z)$ throughout in the unit disc for

$$\lambda_0 = \frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, z_0^{k-1}}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_1^{k-1}} \; \frac{z_1^{k-1} - z_2^{k-1}}{z_1^{k-1} - z_0^{k-1}} .$$

Thus, $h_k(z)$ can not be an extreme point of $F[s, g, \alpha, q, B]$. We next show that f_k and g_k can not be expressed as the convex combination of any two elements of $F(s, g, \alpha, q, B]$. Infact, for z real and positive and $0 \leqslant \lambda \leqslant 1$.

$$f_k(z) < \lambda\left(\frac{(c_k - \alpha d_k) z - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_3^{k-1}}\right)$$

$$+ (1 - \lambda)\left(\frac{(c_k - \alpha d_k) z - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_4^{k-1}}\right)$$

$$z_0 < z_3 \leqslant z_1,\; z_0 < z_4 \leqslant z_1$$

and

$$g_k(z) > \lambda\left(\frac{(c_k - \alpha d_k) z - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_5^{k-1}}\right)$$

$$+ (1 - \lambda)\frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_6^{k-1}}$$

$$z_0 \leqslant z_5 < z_1 \text{ and } z_0 \leqslant z_6 < z_1.$$

The proof is now complete.

Corollary—If $0 < z_0 < z_1 < 1$, the closed conve hull of $F[s, g, \alpha, q, \{z_0, z_1\}]$ is $F[s, g, \alpha, q, [z_0, z_1]]$.

PROOF : Let f_k and g_k be defined as in the theorem. Adopting the method of proof of the theorem it can be shown that the extreme points of $F[s, g, \alpha, q, \{z_0, z_1\}]$ are z, f_k and g_k, $k = 2, 3, \ldots$ Hence the closed convex hull of $F[s, g, \alpha, q, \{z_0, z_1\}]$ is the closed convex hull of $\{z, f_k, g_k : k \geqslant 2\}$. However, it follows from the theorem that the above closed convex hull is $F[s, g, \alpha, q, [z_0, z_1]]$. The proof is complete.

REFERENCES

1. R. M. Goel and N. S. Sohi, *Proc. Am. Math. Soc.* 78 (1983), 353-57.
2. G. P. Kapoor and A. K. Mishra, *Internat. J. Math. Math. Sci.* 7 (1984), 435-42.
3. A. K. Mishra, Ph. D. Thesis, I. I. T. Kanpur, 1981.

4. St. Ruscheweyh, *Comment. Math. Helv.* **52** (1977), 497–509.
5. K. Sakagudi, *J. Math. Soc. Japan* **11** (1959), 72–75.
6. H. Silverman, *Proc. Am. Math. Soc.* **51** (1975), 109–16.
7. H. Silverman, *Trans. Am. Math. Soc.* **219** (1976), 387–95.
8. H. Silverman and E. M. Silvia, *Int. J. Math. Math. Sci.* **2** (1979) 427–39.

Indian J. pure appl. Math., **20** (8) : 829-833, August 1989

A GENERALIZED CARLEMAN BOUNDARY VALUE PROBLEM FOR MULTIPLY CONNECTED DOMAINS

M. G. El Sheikh

Applied Mathematics Department, Faculty of Science, Ain Shams University Cairo, Egypt

(*Received* 19 *August* 1987)

In multiply connected domains D_1, D_2 in the z_1-plane and z_2-plane respectively with Ljapunov boundaries L^j ($i = 1,2$), there is considered the following boundary value problem : To find two functions $\phi_1\,(z_1)$ and $\phi_2\,(z_2)$ analytic in D_1 and D_2 and H-continuous in $D_j + L^j$ according to the boundary condition

$$\phi_2\,[\alpha\,(t)] = a\,(t)\,\phi_1\,(t) + b\,(t)\,\overline{\phi_1\,(t)} + c\,(t) \qquad \ldots(A)$$

where the functions $a\,(t)$, $b\,(t)$ and $c\,(t)$ satisfy H-condition on L^1, and $\alpha\,(t)$ preserves the direction of the circuit on L^1.

The index of problem (A) is calculated, and conditions of its solvability are proved.

Let two complex planes $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be given, and $z_2 = \alpha\,(z_1)$ is the homeomorphism preserving the orientation of the z_1-plane onto the z_2-plane. We take a $(m + 1)$-connected domain D_1 in the z_1-plane bounded by the Ljapunov contour L^1 consisting of closed smooth non-intersecting contour L^1_1, L^1_2, $\ldots$, L^1_m, L^1_{m+1} of which L^1_{m+1} contains all the others. The domain D_2 bounded by smooth closed contours L^2_1, L^2_2, $\ldots$, L^2_m, $L^2_{m\cdot 1}$ not intersecting one another, the last of which encloses all the others, corresponds to the domain D_1 under the transformation $z_2 = \alpha\,(z_1)$.

Assume that a derivative $\alpha'\,(t)$ is different from zero and H-continuous.

We consider the following boundary value problem : To find two functions $\phi_1\,(z_1)$, $\phi_2\,(z_2)$ analytic in D_1 and D_2 and H-continuous in $D_j + L^j$ according to the boundary condition

$$\phi_2\,[\alpha\,(t)] = a\,(t)\,\phi_1\,(t) + b\,(t)\,\overline{\phi_1\,(t)} + c\,(t)\ (t \in L^1) \qquad \ldots(1)$$

where the functions $a\,(t)$, $b\,(t)$, $c\,(t)$ satisfy the Hölder condition on L^1.

The following lemma holds.

Lemma—If $\alpha(t)$ preserves the direction of the circuit on L^1, then the functions $\phi_1(z_1)$, $\phi_2(z_2)$ analytic in D_1 and D_2 can be represented in the form

$$\left.\begin{aligned} \phi_1(z_1) &= -\frac{1}{\pi i}\int_{L^1} \frac{\overline{\phi(\tau)\,d\tau}}{\tau - z_1} - \int_{L^1_m} \overline{\phi(\tau)}\,[1 + |\alpha'(\tau)|]\,d\sigma \quad (z_1 \in D_1) \\ \phi_2(z_2) &= \frac{1}{\pi i}\int_{L^2} \frac{\phi[\beta(\tau)]\,d\tau}{\tau - z_2} + \int_{L^2_m} \phi(\tau)\,[1 + |\alpha'(\tau)|]\,d\sigma \quad (z_2 \in D_2) \end{aligned}\right\} \quad \ldots(2)$$

where σ is the arc coordinate of the point τ on the contour L^1; $\beta(\tau)$ is the inverse of the function $\alpha(t)$; the density $\phi(t)$ is determined to within a constant term of the form $\sum_{k=1}^{m\ 1} \lambda_k \beta_k(t)$, where λ_k are arbitrary complex constants.

Proof : Consider the following integral representation

$$\phi_j(z_j) = \frac{1}{\pi i}\int_{L^j} \frac{\phi_j^*\,d\tau}{\tau - z_j} \quad (z_j \in D_j) \qquad \ldots(3)$$

where the densities satisfy the condition

$$\phi_2^*[\alpha(t)] + \phi_1^*(t) = 0 \qquad \ldots(4)$$

and are defined to within constant terms of the form $\sum_{k=1}^{m} \lambda_k \beta_k(t)$ and $-\sum_{k=1}^{m} \lambda_k \beta_k(t)$, respectively.

By means of the Sokhotski-Plemelj formulae[1] and condition (4) we obtain

$$\begin{aligned} \phi_2(t) - \phi_2^-(t) &= 2\phi_2^*(t) \\ \phi_1(t) - \phi_1^-(t) &= 2\phi_1^*(t) \end{aligned} \qquad \ldots(5)$$

whence

$$\phi_2^-[\alpha(t)] = -\phi_1^-(t) + \phi_2[\alpha(t)] + \phi_1(t) \quad (t \in L^1_k) \qquad \ldots(6)$$

where the function $\phi_j^-(z_j)$, $(j = 1, 2)$ is analytic in the domain which is the comple-

ment of $D_j + L^j$ in the plane and is denoted by D^-_{j1}, D^-_{j2}, ..., $D^-_{j,m+1}$. As the functions $\phi_j(t)$ are given, then it is possible from equation (6) to find the functions $\phi^-_j(t)$ and from equations (5) the required densities $\phi^*_j(t)$. The equality (6) represents the boundary value problem for simply connected domains D^-_{j1}, D^-_{j2}, ..., $D^-_{j,m+1}$.

Let the functions $w = w^{(t)}_k(z_j)$ map conformally the domain D^-_{1k} ($k = 1, 2, \ldots, m+1$) in the z_1-plane and the domain D^-_{2k} in the z_2-plane into the exterior of a unit circle c in the w-plane. By $z_j = V^{(j)}_k(w)$ we denote the inverse of the functions $w = w^{(j)}_k(z_j)$. It is known from the theory of conformal transformation, under the assumed conditions, that not only the functions $w^{(j)}_k(z_j)$, $V^{(j)}_k(w)$ but also their derivatives are continuously continuable on L^j_k and c respectively and satisfy the H-condition.

Introduce the new functions $\psi^-_j(w) = \phi^-_j[V^{(j)}_k(w)]$. It is easily seen that the boundary conditions (6) assume the form

$$\psi^-_2[\alpha_1(\zeta)] = -\psi^-_1(\zeta) + \psi_2[\alpha_1(\zeta)] + \psi_1(\zeta) \quad (\zeta \in c) \qquad \ldots(7)$$

where

$$\alpha_1(\zeta) = w^{(2)}_k\left\{\alpha\,[V^{(1)}_k(\zeta)]\right\}, \quad \psi_j(\zeta) = \phi_j\,[V^{(j)}_k(\zeta)].$$

The function $\alpha_1(\zeta)$ has on c a derivative $\alpha'_1(\zeta)$ different from zero, satisfying the H-condition, and transforms c one-to-one onto itself while preserving the direction of the circuit when $\alpha(t)$ possesses this property on L^1.

Thus, the problems (6) by means of the conformal mapping have been reduced to the problems of Carleman's type for a simply connected domain which are solvable, and the solution of each of them depends linearly on an arbitrary complex constant[2]. The equality (6) in the domain $D^-_{j,m+1}$ assigns the boundary condition of an exterior problem of Carleman's type and is solvable if to find a solution bounded at infinity; here, the constant $\lambda_{m+1} = \lambda$ is determined uniquely by functions $\phi_j(z_j)$. Having defined $\phi_j(z_j)$ in that way and substituted them in (5), we find the densities $\phi^*_j(t)$.

$$\phi^*_1[\alpha(t)] + \phi^*_2(t) = 0$$

and also the densities $\phi_j(t)$ shall be defined to within constant terms of the form

$$\sum_{k=1}^{m} \lambda_k \beta_k(t), \; - \sum_{k=1}^{m} \bar{\lambda}_k \beta_k(t).$$

Putting $\phi_2(t) = \phi[\beta(t)]$, then it is obvious that $\phi_1^*(t) = -\overline{\phi(t)}$, and the function $\phi(t)$ is defined to within a constant term of the form

$$\sum_{k=1}^{m} \lambda_k \beta_k(t).$$

Thus, we have shown that for the functions $\phi_j(z_j)$ there exist the integral representations

$$\left.\begin{aligned} \phi_1(z_1) &= -\frac{1}{\pi i}\int_{L^1} \frac{\phi(\tau)\, d\tau}{\tau - z_1} - \bar{\lambda} \; (z_1 \in D_1) \\ \phi_2(z_2) &= \frac{1}{\pi i}\int_{L^2} \frac{\phi[\beta(\tau)]\, d\tau}{\tau - z_2} + \lambda \; (z_2 \in D_2). \end{aligned}\right\} \qquad (8)$$

We shall show how to obtain formulae (2) from formulae (8). With that end in view, we take any interior contour, for instance L_m^1, and choose a constant δ so as the following equality holds

$$\delta = \int_{L_m^1} \phi(\tau)[1 + |\alpha'(\tau)|]\, d\sigma.$$

Then changing the density $\phi(t) + \delta$ to $\phi(t)$ on the contour L_m^1 and leaving $\phi(t)$ unchangeable on the others, we obtain the representation (2) where the density is defined completely on the contour L_m^1 and on the others to within a constant term of the form

$$\sum_{k=1}^{m-1} \lambda_k \beta_k(t)$$

where λ_k is an arbitrary complex constant.

The lemma has been proved.

With the aid of the integral representation (2) we reduce the boundary value problem (1) to the singular integral equation

$$[1 + b(t)]\phi(t) + a(t)\overline{\phi(t)} + \frac{1}{\pi i}\int_{L^1} \Bigg[\frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)}$$

(equation continued on p. 833)

$$- \frac{b(t)\,\overline{\tau^2(\sigma)}}{\bar{\tau} - \bar{t}} \Big] \phi(\tau)\, d\tau + \frac{a(t)}{\pi i} \int_{L^1} \frac{\overline{\phi(\tau)}\, d\tau}{\tau - t}$$

$$+ [1 + b(t)] \int_{L_m^1} \phi(\tau)\, [1 + |\alpha'(\tau)|]\, d\sigma + a(t) \int_{L_m^1} \overline{\phi(\tau)}$$

$$[1 + |\alpha'(\tau)|]\, d\sigma + a(t) = c(t) \qquad (t \in L^1). \qquad ...(9)$$

The index of this equation over the field of real numbers is equal to 2 ind $b(t)$ Litvinchuk and Hasabov[3]. The boundary condition of the problem adjoint to (1) is written in the form

$$\psi_1(t) = a(t)\,\alpha'(t)\,\psi_2[\alpha(t)] + \overline{b(t)\, t^2(s)\, \alpha(t)\, \psi_2[\alpha(t)]} \; (t \in L) \qquad ..(10)$$

Let l and l' be the numbers of linearly independent solutions of the homogeneous problem (1) and the adjoint problem (10), then it can be shown that

$$l - l' = 2 \text{ ind } b(t) - 2m + 2. \qquad ...(11)$$

For solvability of equation (9) it is neccessary and sufficient that there hold the condition[2]

$$\text{Re} \int_{L^1} c(t)\, \psi_2^{(k)}[\alpha(t)]\, \alpha'(t)\, dt = 0 \;(k = 1, 2, \ldots) \qquad ...(12)$$

where $\left\{ \psi_j^{(k)}(t) \right\}_j$ is the complex system of linearly independent solutions of the adjoint problem (10).

References

1. F. D. Gakhov, *Boundary Value Problems*. Pergamon press, Oxford, 1966.
2. G. S. Litvinchuk and E. G. Hasabov, *Dokl. Akad. SSSR* **140** (1961) 48-51.
3. G. S. Litvinchuk and E. G. Hasabov, *Izv. Vyssh. Uchebn. Zaved. Mathematika*, No. 4, (1964), 99-110 (Russian).

Indian J. pure appl. Math., **20** (8) : 834-839, August 1989

NONSTATIONARY LAW OF HEAT CONDUCTION IN CLASSICAL THERMOELASTIC SOLID

AMIYADEB MUKHERJEE*

N. D. College, Howrah, West Bengal

(*Received* 14 *September* 1987; *after revision* 9 *May* 1989)

The present paper seeks to envisage the classical nonstationary law of heat conduction at finite speed in thermoelastic solid continua. The covariant analysis is carried along purely phenomenological lines without kinetic-theory arguments and based on fundamental concepts of functional-theoretic continuum mechanics. As such, according to the nature of the problem, heat flux vector is considered as a linear functional of suitable well defined functions representable by Volterra-type Riemann convolution in normed Hilbert space which, under proper assumptions, yield the desired nonstationary law as revised version of Fourier's differential equation for anisotropic bodies in acceptable forms. The isotropic case is also treated. It appears, that the results allow transition to the stationary counterparts.

1. INTRODUCTION

Fourier's (linear) law of heat flux is known to render parabolic differential equation predicting infinite speed of thermal propagation in conductors[5]. But, for some reasons (vide section 2 of this text), the feature seems to be undesirable. As way out, notable attempts were made in classical physics, first by Cattaneo[2,3] and next by Vernotte[8], whence nonstationary thermodynamics came into being where hyperbolicity of the modified heat flux equation automatically leads to bounded speed of propagation. Later, the problem has been pursued by other authors with diverse motivations, accounts of which may be had in literatures on the topic (e.g. vide Bressan[1]).

The present studies is also an attempt in this direction, but with different aims and objectives. It seeks a solution to the classical nonstationary problem of modifying Fourier's constitutive law so as to account for heat conduction at finite speed in anisotropic thermoelastic solid continua, which has been hardly paid due attentions erstwhile. The analysis runs along purely phenomenological lines within the framework of axiomatic covariant continuum mechanics and functional theory. The rationale of the paper is as follows.

The heat flux vector is taken as a linear regular functional of well defined appropriate functions which, at some given point, is also a continuous function of time representable by Volterra-type Riemann convolution of tensor and vector-valued

* 69/A, Harish Mukherjee Road, Calcutta 700925.

bounded functions in normed Hilbert space. Finally, simple considerations of integro-differential equations, expansions under suitable assumptions and approximations reduce the integral into a covariant differential equation that appears to be the desired modification of classical Fourier's law for anisotropic solid continuum. Alternative versions of the equation are obtained and some important scalars, tensors and operators related to relaxation and nonstationary processes, are defined. The isotropic case is also discussed. The results appear to be agreeable and allow transition to the stationary counterparts.

2. Solution to Nonstationary Problem of Heat Conduction in Classical Thermoelastic Solid : Modified Fourier's Law without Paradox of Infinite Speed

First, let us consider an elastic solid continuum M in Euclidean space $E_{(3)}$ equipped with arbitrary space coordinates x_α and metric $g_{\alpha\beta}$ (functions of x_α), where (1, 2, 3) is the range of the Greek indices. Next, suppose that owing to some thermal process, M undergoes a deformation described by the strain tensor $e_{\alpha\beta}$ and absolute temperature T, relative to a reference state $R_{(3)}$ at initial temperature $\mathring{T}$ (constant) and time $t = 0$, where $e_{\alpha\beta}$, T are functions of x_α, t.

Then, Fourier's (linear) law of heat conduction stands as[4,6] :

$$q^\alpha = - K_T^{\alpha\beta} \overset{*}{T}_\beta \qquad \text{...(1.1)}$$

q^α being the heat flux vector, $k^{\alpha\beta}$ (positive definite) the conductivity tensor and $\overset{*}{T}_\beta = T_{1\beta}$ the temperature gradient which, since $g^\alpha_{\beta|\alpha} = 0$, may also be described by the equation

$$\overset{*}{T}_\beta = \overset{*}{T}_\alpha\, g^\alpha_\beta = \left(T g^\alpha_\beta \right)_{|_\alpha} = \delta^y_\alpha \left(T g^\alpha_\beta \right)_{|_\lambda} \qquad \text{...(1.1')}$$

where δ^y_α is the (mixed) Kronecker delta and the vertical stroke (|) denotes covariant derivative in $E_{(3)}$. Herein, we observe that $e_{\alpha\beta}$, $g_{\alpha\beta}$, $k^{\alpha\beta}$ and all other tensors to appear, are assumed to be symmetric in their covariant (or contravariant) indices and dummy suffix means summation.

Now, as it is well known, the relation (1.1) leads to parabolic differential equation which predicts infinite speed of heat conduction[4]. Incidentally, transmission of elastic waves (e.g. in incompressible fluid) and strains also offer similar examples. The results seem to be sufficient provided of course we treat systems, which are mechanically constrained, as first approximative ones. But, from standpoint of a more rigorous nature, propagation without bound is an undesirable feature in every such process which takes place in a material continuum. As such, we propose to revise equation (1.1) so that propagation of thermal disturbances in M is at finite speed.

To this end, it seems proper to compare (1.1) with the Hooke's law for stress-strain of similar type in elasticity theory and to note that the later also indicates instantaneous strain[7], which can be avoided through functional approach based on hereditary principles where relaxation phenomenon suppresses the infinite speed of propagation[5,9].

Likewise, as the above example suggests and also as it is justifiable from the axioms of determinism and memory in modern continuum mechanics[5], let us accept heat conduction as a hereditary—type event with fading memory so that the process is without the paradox of infinite speed[5].

With this point in view towards modifying a linear law, we propose to define q^{α} (response) as a Vector-valued first degree bounded functional of Q_{α} (action) which is a a vector assumed to be regular at the time-origin and some suitable function of x_{λ}, T, $\overset{*}{T}_{\lambda}$, $\epsilon_{\lambda\delta}$ such that heat flux q^{α} is also a continuous differentiable function of time over permissible domain. Now, collection of all histories of a physical process of fading memory is a Hilbert space where bounded linear functionals have defined norms and inner products. Hence, given x_{μ}, it is further assumed that q^{α} is a linear function in the Hilbert space of tensor- and vector-valued time-functions e.g. $\zeta(t)$, $\eta(t)$ on $[0, \infty)$ having finite norms $\|F\|^2 = \int_0^{\infty} |F(t)|^2 f(t)\,dt$, $(F = \zeta, \eta)$, and the inner product as required[4,10]:

$$<\zeta, \eta> = <\zeta(0), \eta(0)> + \int_0^{\infty} <\zeta(t), \eta(t)> f(t)\,dt \qquad \ldots(1.2)$$

with, obliviator (non-negative)

$$f(t) \to 0, \text{ as } t \to \infty. \qquad \ldots(1.2')$$

Then, as per representation of linear forms in Hilbert space consistent with the problem, we can, by proper choice of functions, write q^{α} as a Volterra-type Riemann convolution at time t and at point x_{μ} of M in $E_{(3)}$:

$$q^{\alpha}(t) = \int_0^t \overset{*}{d}{}^{\alpha\beta}(t, t')\, Q_{\beta}(t')\,dt';\ t' \in [0, t],\ t \in [0, \infty) \qquad \ldots(1.3)$$

with,

$$d^{\alpha\beta}(t, t') = f^{\alpha\beta} e^{-(t-t')/k'};\ \overset{*}{d}{}^{\alpha\beta}(t, t') = \frac{\partial d^{\alpha\beta}(t, t')}{\partial t'}$$

$$= \frac{f^{\alpha\beta}}{k'} e^{-(t-t')/k'} \qquad \ldots(1.3')$$

$$q^{\alpha}(t') \equiv Q_{\alpha}(t'),\ t' \leqslant 0;\ \text{also } q^{\alpha} \equiv Q_{\alpha} \equiv 0, \text{ when } \overset{*}{T}_{\beta} = 0 \qquad \ldots(1.3'')$$

where, the tensors $d^{\alpha\beta}$ and $f^{\alpha\beta}$ (time-independent) are each assumed to be positive definite, and k' is a positive scalar (parameter) of the order of time such that for large t and fixed t', the expression $\exp(-t/k')$ characterizes rapid decay of $d^{\alpha\beta}(t, t')$ which is also small when $t \gg k'$. Thereby, $d^{\alpha\beta}$ plays the role of relaxation tensor and k' the relaxation time which checks infinite speed of propagation.

The equation (1.3) with (1.3′) - (1.3″) then defines, in integral from, a modified version of the classical Fourier's law (1.1) for heat conduction at finite speed in the continuum M.

Next, we observe that though the kernel $\overset{*}{d}{}^{\alpha\beta}(t, t')$ is not a polynomial in t, it is degenerate and such that, by (1.3′), $\overset{*}{d}{}^{\alpha\beta}(t, t) \equiv f^{\alpha\beta}/k$ (a nonzero time-independent quantity) has a 0-fold root for $t = 0$. Hence, in order that a solution of (1.3) may exist and be finite at the origin, we should necessarily have an 1-fold root for $q^{\alpha}(t)$ at $t = 0$. Then, Q_{β} must satisfy the following first order differential equation obtained by differentiating (1.3) once with respect to t and eliminating the integral :

$$q^{\alpha} + k' \dot{q}^{\alpha} = f^{\alpha\beta} Q_{\beta} \qquad \text{...(1.4)}$$

with

$$\dot{q}^{\alpha} = Dq^{\alpha}(t);\ D \equiv \frac{d}{dt} = v_{\alpha} \frac{\partial}{\partial x_{\alpha}} + \frac{\partial}{\partial t}\left(\simeq \frac{\partial}{\partial t}, \text{ for small motions}\right),$$

$$v_{\alpha} = \frac{dx^{\alpha}}{dt} \qquad \text{...(1.4′)}$$

where $\dot{q}^{\alpha}$ may be taken as a vector in $E_{(3)}$; since Q_{β} is regular at origin and $q^{\alpha}(0) \equiv Q_{\alpha}(0) \equiv 0$, the differential equation (1.4) yields a solution which is also an unique solution of the integral equation (1.1)[7,9].

As for the arbitrary function Q_{β}, we note that linear expansion yields,

$$Q_{\beta} = \mathring{Q}_{\beta} - h^{\lambda}_{\beta} \overset{*}{T}_{\lambda} + \omega^{\lambda\delta}_{\beta} e_{\lambda\delta} \qquad \text{...(1.5)}$$

where $\mathring{Q}_{\beta}$, $h^{\beta\lambda}$ (positive definite), and $\omega^{\lambda\delta}_{\beta}$ depends on T, x_{μ} only. But, since $q^{\beta} \equiv Q_{\beta} \equiv 0$ when $\overset{*}{T}_{\lambda} = 0$, we must have $\mathring{Q}_{\beta} \equiv 0$ and $\omega^{\lambda\delta}_{\beta} \equiv 0$ irrespective of $e_{\delta\lambda}$; for, otherwise the tensorial equation (1.5) would become inconsistent having a scalar zero (a tensor of order zero) and a nonzero vector (a tensor of order one) on either sides of it. Hence,

$$Q_{\beta} = -h^{\lambda}_{\beta} \overset{*}{T}_{\lambda} \qquad \text{...(1.5′)}$$

so that

$$f^{\alpha\beta} Q_\beta = - \kappa^{\alpha\lambda} \overset{*}{T}_\lambda \qquad ...(1.5'')$$

with

$$k^{\alpha\lambda} = - f^{\alpha\beta} h^\lambda_\beta \qquad ...(1.5''')$$

where the form shown by (1.5″) must be quadratic due to the minus sign before $k^{\alpha\beta}$ (positive definite) and on account of $- q^\alpha \overset{*}{T}_\alpha \geqslant 0$ in keeping with the fact that heat flows from higher to lower temperature zones in M^6.

Taking (1.4) and (1.5″), we finally arrive at the result

$$q^\alpha + k' \dot{q}^\alpha = - k^{\alpha\beta} \overset{*}{T}_\beta \qquad ...(1.6)$$

where $k^{\alpha\beta}$ is identified as the thermal conductivity tensor, for (1.6) reduces to (1.1) when $k' = 0$.

The equation (1.6) is then a covariant version of (1.3), which constitutes a possible modification of the equation (1.1) for heat propagation in M without the paradox of infinite speed. Note that the other equation in q_α would be

$$q_\alpha + k' \dot{q}_\alpha = - k^\beta_\alpha \overset{*}{T}_\beta. \qquad ...(1.6a)$$

If inner product with the metric tensors be performed, we can also write the respective differential equations (1.6) and (1.6a) in operational notations as follows :

$$D^\alpha_\beta \, q^\beta \equiv D^{\alpha\beta} q_\beta = - k^{\alpha\beta} \overset{*}{T}_\beta \qquad ...(1.7)$$

and

$$D_{\alpha\beta} \, q^\beta \equiv D^\beta_\alpha \, q_\beta = - k^\beta_\alpha \overset{*}{T}_\beta \qquad ...(1.7a)$$

with

$$D_{\alpha\beta} = g_{\alpha\beta} + g_{\alpha\beta} k' D; \; D^{\alpha\beta} = g^{\alpha\beta} + g^{\alpha\beta} k' D; \; D^\alpha_\beta = g^\alpha_\beta + g^\alpha_\beta k' D \qquad ...(1.7)$$

where $D_{\alpha\beta}$ may be called the classical covariant relaxation operator.

In particular, for an isotropic model, we may set $k^{\alpha\beta} = k\, \delta^{\alpha\beta}$ where the positive scalar k is the thermal conductivity. Then (1.7)—(1.7a) respectively reduce to

$$D^\alpha_\beta \, q^\beta \equiv D^{\alpha\beta} q_\beta = - k \, \delta^{\alpha\beta} \overset{*}{T}_\beta \qquad ...(1.8)$$

and

$$D^{\beta}_{\alpha}\, q_{\beta} \equiv D_{\alpha\beta}\, q^{\beta} = -\,\kappa\, \delta^{\beta}_{\alpha}\, \overset{*}{T}_{\beta} \qquad \text{...(1.8a)}$$

which agree in essence with the result obtained by Cattaneo[2] and Vernotte[8] from different approaches, assumed that $D \simeq \partial/\partial t$ for small motions.

3. Conclusion

Looking back, we find that it is possible to obtain the classical nonstationary law of heat conduction in thermoelastic solid continua on phenomenological lines, by proper choice of assumptions, mathematical techniques, and right type of independent arguments for dependent variables consistent with the basic principles of functional theory and tensorial continuum mechanics. Accordingly, heat flux vector is considered as a bounded linear functional of space coordinates, absolute temperature, temperature gradient and strain tensor, being itself a differentiable function of time representable by Volterra-type Riemann convolution integral in Hilbert space, which finally yields modified classical covariant version of Fourier's differential equation for thermal propagation without infinite speed. The isotropic case is also treated where the results are found to agree in essence with those obtained by others (e.g. vide[2,5,8]). In particular, the stationary case also appears to be recoverable.

Acknowledgment

The author is thankful to Professor D. K. Sinha, formely of Jadavpur University, for his kind supervision and to the U. G. C., Jadavpur University and N. D. College (West Bengal, India) for necessary supports in the work carried out under F. I. P. of the U. G. C.

References

1. A. Bressan, *Atti Accad. Naz. Lincei* **17**, (1983), 59–100.
2. C. Cattaneo, *Atti Semin. Mat. Fis. Univ. Modena* **3** (1948), 3–21.
3. C. Cattaneo, *C. R. Acad. Sci. (Paris)* **A247** (1958), 431–33.
4. A. C. Eringen, Continuum Physics (Ed.), Pts. I & II, *Vol. 12*, Academic Press, N. Y., 1975.
5. G. Maugin, *J. Phys.* **A7** (1974), 465–84.
6. W. Nowacki, *Dynamic Problems of Thermoelasticity*, Noordhoff Int. Pub., Netherlands, 1975.
7. Yu. N. Rabotnov, *Elements of Hereditary Solid Mechauics*, Mir Pub., Moscow (1980).
8. F. Vernotte, *C. R Acad Sci (Paris)* **A246** (1958), 3154–55.
9. V. Volterra, *The Theory of Functionals and of Integral And Integro-Differential Equations*, Dover Pub. Inc., N. Y. (1959).
10. K. Yosida, *Functional Analysis* (Fourth Edition; Springer International Student Edition) Narosa Pub. House, New Delhi, India (1974).

Indian J. pure appl. Math., **20** (8) : 840–849, August **1989**

ELLIPSOIDAL INCLUSIONS IN AN ELASTIC MEDIUM

A. D. ALAWNEH AND N. T. SHAWAGFEH
Mathematics Department, University of Jordan, Amman-Jordan

(*Received* 6 *June* 1988)

The problem of classical elasticity where a homogeneous linear ellipsoidal solid is embedded in an unbounded, homogeneous, isotropic, elastic medium is discussed. Exact solutions for the disturbance displacement field due to translational and rotational modes of the solid are obtained by distributing some types of concentrated singularities over the focal ellipse. These solutions extend earlier solutions for rigid spheroids.

1. INTRODUCTION

Many boundary value problems in different branches of mathematical physics are resolved by the distribution of singularities on, along or over some part of the body being investigated. This method, known as the singularity method, has proved to be effective in wide range of configurations in many fields such as low-Reynolds number hydrodynamics[1,2], potential theory[3], and scattering theory[4]. In elastostatics, Kanwal and Sharma[5] first use this method to investigate the displacement problems involving rigid spheroid translating or rotating about, one of its axis. The problem of translating thin oblate bodies has been solved by one of the authors[6].

Our aims in this paper is to extend this method to solve some displacement boundary value problems involving rigid ellipsoidal bodies. In section 2 we present the singularities needed in this analysis and in the rest of the paper we obtain the exact solutions for two modes of displacing an embedded rigid triaxial ellipsoid, first, when the ellipsoid is given a linear displacement and secondly when it rotates about any given axis. In these problems the displacement field is consturcted as superposition of suitable singularities distributed over the focal ellipse. Applying the boundary condition on the surface we obtain integral equations for the density of these distributions which are solved exactly.

Physical quantities such as the force and the torque are obtained. Other quantities such as the stress and strain tensors can be derived directly using their relations with the displacement field[7]. The results for spheroid and disk are obtained as degenerate ellipsoids.

2. SINGULAR SOLUTIONS

Consider a solid with surface S embedded in an unbounded, homogenuous,

linearly elastic and isotropic medium, then the displacement field $\bar{u}$ satisfies the equilibrium equation

$$\mu\left[\frac{1}{1-2\nu}\nabla(\nabla.\bar{u})+\nabla^2\bar{u}\right]+\bar{f}=0 \qquad \ldots(2.1)$$

where μ is the shear modulus, ν the Poisson's ratio of the material and $\bar{f}$ the body force per unit volume.

The primary fundamental solution (singularity) of (2.1) corresponding to a concentrated force

$$\bar{f}=16\,\pi\mu\,(1-\nu)\,\delta\,(x)\,\bar{\alpha} \qquad \ldots(2.2)$$

where $\delta\,(\bar{x})$ is the Dirac delta function, and $\bar{\alpha}$ represents the direction and the magnitude of the force, is called the Kelvin solution and it is given by[7]

$$\bar{U}^k\,(x;\alpha)=\frac{(3-4\nu)\,\alpha}{r}+\frac{(\alpha.\,\bar{x})\,\bar{x}}{r^3},\ r=|\,\bar{x}\,|. \qquad \ldots(2.3)$$

Another important singularity is called the doublet which is formed by two centres of dilatation[7] and it is given by

$$\bar{U}^k\,(\bar{x};\bar{\alpha})=\nabla\,(\bar{\alpha}\ \nabla)\ \frac{1}{r}. \qquad \ldots(2.4)$$

Because of linearity of (2.1) any derivative of (2.3) and (2.4) is also a singular solution.

The first derivative of (2.3) in the direction of $\bar{\beta}$ is called the Kelvin doublet

$$\bar{U}^{kd}\,(\bar{x};\bar{\alpha},\bar{\beta})=-(\bar{\beta}.\,\nabla)\,\bar{U}^k\,(\bar{x};\bar{\alpha}). \qquad \ldots(2.5)$$

The stresslet is the symmetric part of (2.5), that is,

$$\bar{U}^s\,(\bar{x};\bar{\alpha},\bar{\beta})=\tfrac{1}{2}\,[\bar{U}^{kd}\,(\bar{x};\bar{\alpha},\bar{\beta})+\bar{U}^{kd}\,(\bar{x};\bar{\beta},\bar{\alpha})]. \qquad \ldots(2.6)$$

The antisymmetric part is called centre of rotation and it is given by

$$\bar{U}^r\,(\bar{x};\bar{\gamma})=\nabla\times\frac{\bar{\gamma}}{r},\ (\bar{\gamma}=\bar{\alpha}\times\bar{\beta}). \qquad (2.7)$$

The first derivative of (2.4) in the direction of $\bar{\beta}$ is known as quadrupole

$$\bar{U}^{dd}\,(\bar{x};\bar{\alpha},\bar{\beta})=-(\bar{\beta}.\nabla)\,\bar{U}^d\,(\bar{x};\bar{\alpha}). \qquad \ldots(2.8)$$

For a control volume V enclose each of these singularities, the stresslet and quadrupole contribute neither a force nor a torque, while the centre of rotation exerted a torque

$$M=-8\,\pi\mu\,\bar{\gamma}. \qquad \ldots(2.9)$$

3. Statement of the Problem

Let S be the surface of the triaxial rigid ellipsoid

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = 1, \; a_1 \geqslant a_2 \geqslant a_3 > 0 \qquad \text{...(3.1)}$$

then the displacement field satisfies the homogenuous equations

$$\frac{1}{1-2\nu} \nabla (\nabla . \bar{u}) + \nabla^2 \bar{u} = 0 \qquad \text{...(3.2)}$$

with the boundary conditions

$$\bar{u} = \bar{U} \text{ on } S \qquad \text{...(3.3)}$$

and

$$\bar{u} \to 0, \text{ as } | \bar{x} | \to \infty \qquad \text{...(3.4)}$$

where $\bar{U}$ is known and represents the mode of displacement.

We construct the solution of the above boundary value problem by distributing some of the singular solutions of section 2 over the interior of the focal ellipse $E(\bar{y})$ which is defined as the degenerate elliptical disk in the family of ellipsoids confocal to S. Its equation is

$$\frac{y_1^2}{h_1^2} + \frac{y_2^2}{h_2^2} = 1, \; y_3 = 0. \qquad \text{...(3.5)}$$

where the major and minor semi axes of ϵ are defined by

$$h_i^2 = a_i^2 - a_3^2, \qquad i = 1, 2. \qquad \text{...(3.6)}$$

The role of E in potential theory of ellipsoids has been investigated by Miloh[8].

The choice of the appropriate singularities depends on the form of $\bar{U}$, that is, on the mode of displacement involved. In low-Reynolds number flow problems some guidance rules are proposed[1,2] and those are found to be useful for many ptoblems in other fields including elastostatics[5,6].

We shall now consider two types of modes of displacement, translation and rotation of ellipsoid.

4. Translation of Ellipsoid

Let the ellipsoid S be given a linear displacement

$$\bar{U}(\bar{x}) = \bar{V}, \; (\bar{V} \text{ is constant}) \qquad \text{...(4.1)}$$

then we construct the solution of (3.2) by distributions of Kelvins solutions and doublets over E directed along the cartesian axes, $\hat{e}_i$, that is,

$$\bar{u}(\bar{x}) = \sum_{i=1}^{3} \iint_E [A_i q^{-1} \bar{U}^k (\bar{x} - \bar{y}; \hat{e}_i) + B_i q \bar{U}^d (\bar{x} - \bar{y}; \hat{e}_i)] \, dA \quad \text{...(4.2)}$$

where

$$\bar{x} = (x_1, x_2, x_3), \; y = (y_1, y_2, 0), \; dA = dy_1 \, dy_2 \quad \text{...(4.3)}$$

and

$$q = \left[1 - \frac{y_1^2}{h_1^2} - \frac{y_2^2}{h_2^2}\right]^{1/2}. \quad \text{...(4.4)}$$

The constants A_i and B_i represents the strength of the singularities and are to be determined. The function q plays a prominent role in the potential theory of ellipsoid particle[8].

The integrations over E can be carried out by making use of the following integral identities.

$$\iint_E \frac{y_i^{n+1}}{r} q^m \, dA = \frac{h_i^2}{m+2}\left[n \iint_E \frac{y_i^{n-1}}{r} q^{m+2} \, dA - \frac{\partial}{\partial y} \iint_E \frac{y_i^n}{r} q^{m+2} \, dA\right] m \neq -2 \quad \text{...(4.5)}$$

$$\iint_E \frac{q^{2n-1}}{r} \, dA = \frac{(-1)^n (2n)! \, \pi \, h_1 h_2}{2^{2n} (n!)^2} \int_\lambda^\infty \left[\sum_{i=1}^{3} \frac{X_i^2}{a_i^2 + t} - 1\right]^n \frac{dt}{D(t)} \quad \text{...(4.6)}$$

where

$$D(t) = \prod_{i=1}^{3} (a_i + t)^{1/2} \quad \text{...(4.7)}$$

and λ is the positive root of the equation

$$\sum_{i=1}^{3} \frac{X_i^2}{a_i^2 + \lambda} = 1. \quad \text{...(4.8)}$$

The proof of (4.5) is direct through integeration by parts and formula (4.6) has been established by Kim[2]. We note that on the surface of the ellipsoid $\lambda = 0$.

After some calculations the representation (4.2) can be written in the integerated form,

$$\bar{u}\,(\bar{x}) = \pi\, h_1\, h_2 \sum_{i=1}^{3} \Big([A_i \{(3 - 4\nu)\, I\,(\lambda) + (a_i^2 - a_3^2)\, I_i\,(\lambda)\} - B_i\, I_i\,(\lambda)]\, \hat{e}_i - [A_i \{\nabla I\,(\lambda) - (a_i^2 - a_3^2)\, \nabla I_i\,(\lambda)\} - B_i\, \nabla I_i\,(\lambda)]\, x_i\Big) \qquad \ldots(4.9)$$

where

$$I\,(\lambda) = \int_{\lambda}^{\infty} \frac{dt}{D\,(t)},\quad I_i\,(\lambda) = \int_{\lambda}^{\infty} \frac{dt}{(a_i^2 + t)\, D\,(t)} \qquad \ldots(4.10)$$

these integrals satisfy the relations

$$\sum_{i=1}^{3} I_i\,(\lambda) = \frac{2}{a_1\, a_2\, a_3} \qquad \ldots(4.11)$$

$$\sum_{i=1}^{3} a_i^2\, I_i\,(\lambda) = I\,(\lambda). \qquad \ldots(4.12)$$

The solution $\bar{u}\,(\bar{x})$ satisfies the condition at infinity (3.4) since each singularity does. Applying the surface condition (3.3) we obtain

$$\bar{V} = h_1\, h_2 \sum_{i=1}^{3} \Big([A_i \{(3 - 4\nu)\, I + (a_i^2 - a_3^2)\, I_i\} - B_i\, I_i]\, \hat{e}_i - [(a_3^2\, A_i + B_i)\, x_i\, \nabla\, I_i]\Big). \qquad \ldots(4.13)$$

where

$$I = I\,(0) \text{ and } I_i = I_i\,(0). \qquad \ldots(4.14)$$

Equation (4.13) is satisfied if

$$A_i = -\frac{1}{a_3^2}\, B_i = \frac{V_i}{\pi h_1 h_2} \{(3 - 4\nu)\, I + a_i^2\, I_i\}^{-1}. \qquad \ldots(4.15)$$

The net force experienced by the ellipsoid is obtained by adding the contributions of the forces (2.2) over the focal ellipse, hence,

$$\bar{F} = 16\, \pi\mu\, (1 - \nu) \sum_{i=1}^{3} \Big(\iint_E A_i\, q^{-1}\, dA\Big)\, \hat{e}_i \qquad \ldots(4.16)$$

$$\bar{F} = 32\,\pi\mu\,(1 - \nu) \sum_{i=1}^{3} [(3 - 4\nu)\, I + a_i^2\, I_i]^{-1}\, V_i\, \hat{e}_i. \qquad \ldots(4.17)$$

This complete the solution.

5. Rotation of Elliposid

Suppose that the elipsoid S rotates with angular velocity $\bar{\Omega}$ then the boundary condition on the surface is

$$\bar{u}\,(\bar{x}) = \bar{U} = \bar{\Omega} \times \bar{x}, \bar{x} \in S. \qquad \ldots(5.1)$$

In this case we write the displacement field as

$$u\,(x) = \sum_{i=1}^{3} \iint_E C_{iq}\, \bar{U}^r\,(\bar{x} - \bar{y}, \hat{e}_i)\, dA$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}} \iint_E [q\, S_{ij}\, \bar{U}^s\,(\bar{x} - \bar{y};\, \hat{e}_i, \hat{e}_j)$$

$$+ q^3\, Q_{ij}\, \bar{U}^{dd}\,(\bar{x} - \bar{y};\, \hat{e}_i, \hat{e}_j)]\, dA \qquad \ldots(5.2)$$

where C_i, S_{ij} and Q_{ij} are constants to be determined. Due to symmetry we have

$$S_{ij} = S_{ji}, \; Q_{ij} = Q_{ji}. \qquad \ldots(5.3)$$

Performing the integrations in (5.2) using the identities (4.5) and (4.6) we obtain

$$\iint_E q\, \bar{U}^r\,(\bar{x} - \bar{y}; \hat{e}_i)\, dA = -\pi\, h_1\, h_2\, \hat{e}_i \times \sum^{3} x_k\, I_k\,(\lambda)\, \hat{e}_k \qquad \ldots(5.4)$$

$$\iint q\, \bar{U}^s\,(\bar{x} - \bar{y}; \hat{e}_i, \hat{e}_i)\, dA = -\pi\, \frac{h_1\, h_2}{2}\, [\, \{(3 - 4\nu)\, I_j\,(\lambda) - I_i\,(\lambda)\}\, x_j \hat{e}_i$$

$$+ \{(3 - 4\nu)\, I_i\,(\lambda) - I_j\,(\lambda)\}\, x_i\, \hat{e}_j - x_i\, x_j\, \{\nabla I_i\,(\lambda) + \nabla I_j\,(\lambda)\}$$

$$+ \sum_{k=1}^{2} h_k^2\, [\{x_j\, I_{kj}\,(\lambda)\, \delta_{ik} + x_i\, I_{ki}\, \delta_{jk}\}\, \hat{e}_k$$

$$+ \{I_{kj}\,(\lambda)\, \delta_{ik}\, \hat{e}_j + I_{ki}\, \delta_{jk}\, \hat{e}_i + x_j\, \delta_{ik}\, \nabla I_{kj}\,(\lambda) + x_i\, \delta_{jk}\, \nabla I_{ki}\,(\lambda)\}$$

$$\times\, x_k] \qquad \ldots(5.5)$$

and

$$\iint_E q^3\, \bar{U}^{dd}(\bar{x} - \bar{y}; e_i, \hat{e}_j)\, dA$$

$$= -\, 3\pi\, h_1\, h_2\, [(x_i\, \hat{e}_j + x_j\, \hat{e}_i)\, I_{ij}(\lambda) + x_i\, x_j\, \nabla I_j(\lambda)] \qquad \text{...(5.6)}$$

where

$$I_{ij}(\lambda) = \int_\lambda^\infty \frac{dt}{(a_i^2 + t)\,(a_j^2 + t)\, D(t)} \qquad \text{...(5.7)}$$

which satisfy the identity

$$(a_i^2 - a_j^2)\, I_{ij} = I_j - I_i. \qquad \text{...(5.8)}$$

Applying the boundary condition (5.1) and evaluating (5.2) on the surface, we obtain after a straight forward calculations

$$C_1 = \Omega_1\, J_{23}\, K_{23},\; C_2 = \Omega_2\, J_{13}\, K_{13},\; C_3 = \Omega_3\, J_{12}\, K_{12} \qquad \text{...(5.9)}$$

$$S_{12} = (-3/a_3^2)\, Q_{12} = \Omega_3\, H_{12}\, K_{12} \qquad \text{...(5.10)}$$

$$S_{13} = (-3/a_2^2)\, Q_{13} = \Omega_2\, H_{13}\, K_{13} \qquad \text{...(5.11)}$$

$$S_{23} = (-\,3/a_1^2)\, Q_{23} = \Omega_{23} = \Omega_1\, H_{23}\, K_{23} \qquad \text{...(5.12)}$$

where

$$J_{ij} = 2\,\{(1 - \nu)\, a_j^2 + \nu\, a_i^2\}\, I_i - 2\,\{(1 - \nu)\, a_i^2 + \nu a_j^2\}\, I_j \qquad \text{...(5.13)}$$

$$K_{ij} = \frac{1}{\pi h_1\, h_2}\, \{a_i^2\, I_i - a_j^2\, I_j^2 - (3 - 4\nu)\,(a_i^2 - a_j^2)\, I_i\, I_j\}^{-1} \qquad \text{...(5.14)}$$

and

$$H_{ij} = 2\,(a_j^2 - a_j^2)\,(I_i - I_j). \qquad \text{...(5.15)}$$

The torque exerted on the ellipsoid is obtained by adding the contributions of the centers of rotation and using formula (2.9), thus

$$\bar{M} = -\, 8\,\pi\,\mu \sum_{i=1}^{3} \left(\iint_E C_i\, q\, dA\right) \hat{e}_i \qquad \text{...(5.16)}$$

$$\bar{M} = -\,\frac{16}{3}\,\pi^2\,\mu\, h_1 h_2 \sum^{3} C_i\, \hat{e}_i. \qquad \text{...(5.17)}$$

In the limit $\nu \to 1/2$ and $2\mu\nu/(1-2\nu)\ \nabla.\bar{u} \to -P$, the elastostatic equation (3.2) reduces to Stokes equations of fluid mechanics, where $\bar{u}$ denotes the velocity field and P denotes the pressure. In this case formulas (4.17) and (5.17) reduces to the drag and torque formulas given in Jeffery[9].

6. Degenerate Ellipsoids

(a) *Spheroid*

When $a_2 = a_3$ the ellipsoid (3.1) degenerates into the prolate spheroid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2 + x_3^2}{a_2^2} = 1. \qquad \ldots(6.1)$$

In this case all the ellipsoid integrals in the previous analysis are elementary and are given by

$$I = L/(ea_1) \qquad \ldots(6.2)$$

$$I_1 = \frac{1}{e^3\, a_1^3}\ (-2e + L) \qquad \ldots(6.3)$$

$$I_2 = I_3 = \frac{1}{2e^3\, a_1^3}\ (-2e/(1-e^2) + L) \qquad \ldots(6.4)$$

where $L = \ln\left(\dfrac{1+e}{1-e}\right)$, and e is the eccentricity of the spheroid defined by

$$e^2 = 1 - a_2^2/a_1^2. \qquad \ldots(6.5)$$

Substituting these into (4.17) and (5.17) we obtain the force and the torque exerted by the prolate spheroid. Their components are

$$F_1 = e^3\, a_1\, V_1\, [2e + (3e^2 - 4\nu e^2 - 1)\, L]^{-1} \qquad \ldots(6.6)$$

$$\frac{F_2}{V_2} = \frac{F_3}{V_3} = 2e^3\, a_1\, [-\,2e^2 - (1 - 7e^2 + 8\nu e^2)\, L]^{-1} \qquad \ldots(6.7)$$

$$M = -\,(32/3)\,\pi\,\mu\,\Omega_1\, e^3\, a_1\, [(2e/1-e) - L]^{-1} \qquad \ldots(6.8)$$

and

$$\frac{M_2}{\Omega_2} = \frac{M_3}{\Omega_3} = -\,\frac{64}{4}\,\pi\mu\, e^3\, a^3\, [2e\,\{3 - (4-\nu)\, e + 2\,(1-\nu)\, e^4$$

$$-\,(1-e^2)\,(3 - 2e^2 + \nu e^2)\, L]\,\Delta^{-1} \qquad \ldots(6.9)$$

where $\Delta = 4e^2\,\{3 + 2e^2 - 8\nu e^2\} - 4e\,\{3 + (3-8\nu)\, e^2 - (3-4\nu)\, e^4\}\, L$
$$+\,(1-e^2)\,\{3 + (7-8\nu)\, e^2\}\, L^2.$$

The corresponding results for oblate spheroid can be obtained by replacing e by $ie(1-e^2)^{1/2}$. Formula (6.8) agrees with the one obtained by Kanwal and Sharma[5] for

the torque exerted by a prolate spheroid rotating about its major axis and (6.9) coincide with that for rotation about the minor axis.

(b) *Disks*

When $a_3 \to 0$, the ellipsoid degenerates into an elliptic disk whose axis lies along the x_3-axis. The elliptic integrals in this limit become

$$I = \frac{2}{a_1} K \quad \ldots(6.10)$$

$$I_1 = \frac{2}{a_1^3 k^2} (K - E) \quad \ldots(6.11)$$

$$I = \frac{2}{a_1^3 k^2 k'^2} (E - k'^2 K) \quad \ldots(6.12)$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, with argument

$$k^2 = 1 - a_2^2/a_1^2 \quad \ldots(6.13)$$

and k'^2 is defined by the relation

$$k^2 + k'^2 = 1. \quad \ldots(6.14)$$

The components of the force experienced by the translating elliptic disk are

$$F_1 = -16\pi\, a_1 k^2 (1 - \nu) V_1 [\{1 + (3 - 4\nu) k^2\} K - E]^{-1} \quad \ldots(6.15)$$

$$F_2 = 16\,\pi\, a_1 k^2 (1 - \nu) V_2 [\{(1 - 4 (1 - \nu) k^2\} K - E]^{-1} \quad \ldots(6.16)$$

and

$$F_3 = 16\,\pi\, a_1 \mu (1 - \nu) V_3 \{(3 - 4\nu) K\}^{-1} \quad \ldots(6.17)$$

and the components of the torque exerted by a rotating elliptic disk are

$$M_1 = -(32/3)\pi\mu (1 - \nu)\, \Omega_1 \{(3 - 4\nu) I_2\}^{-1} \quad \ldots(6.18)$$

$$M_2 = -(32/3)\,\pi\mu (1 - \nu)\, \Omega_2 \{(3 - 4\nu) I_1\}^{-1} \quad \ldots(6.19)$$

and

$$M_3 = -\frac{32}{3}\pi\mu\,\Omega_3 [\{(1 - \nu) a_2^2 + \nu a_1^2\} I_1 - \{(1 - \nu) a_2^2 + \nu a_2^2\} I_2]$$
$$+ [a_1^2 I_1^2 - a_2^2 - (3 - 4\nu)(a_1^2 - a_2^2) I_1 I_2]^{-1}. \quad \ldots(6.20)$$

The limiting case of circular disk ($a_1 \to a_2$) yields

$$\frac{F_1}{V_1} = \frac{F_2}{V_2} = -\frac{64\, a_1 \mu (1 - \nu)}{7 - 8\nu},\; F_3 = \frac{32 a_1 \mu (1 - \nu)}{3 - 4\nu} \quad \ldots(6.21)$$

$$\frac{M_1}{\Omega_1} = \frac{M_2}{\Omega_2} = \frac{64a_1^3 \mu (1 - \nu)}{3 (3 - 4\nu)}, \; M_3 = - \frac{32a_1^3\mu}{3}. \qquad \text{...(6.22)}$$

Formulas (6.21) and (6.22) agree with results in Lur'e[10].

7. Conclusion

In this paper we extend the method of singularities to solve two problems of clastostatics involving translational and rotational ellipsoidal bodies and their degenerate cases. Compared with the usual separation of variables method we note that the singularity solutions are far more simple in from since it does not depend on the choice o f the appropriate coordinate system and therefore instead of using ellipsoidal harmonics expansions to represent the solution, it is now represented in terms of simple elliptic integrals. The only difficulty in using the singularity method is the choice of the proper singularities for each mode of displacement. It is hoped that by this method more boundary value problems in elasticity are to be solved, specifically the stress type problems. Extension of the singularity method for elastodynamic problems of ellipsoid are under investigation.

References

1. A. T. Chwang and T. Y. Wu, *J. Fluid Mech.* **67** (1975), 787–815.
2. S. Kim and P. V. Arunachalam, *J. Fluid Mech.* **178** (1987), 535–47.
3. A. D. Alawneh and R. P. Kanwal, *SIAM Rev.* **19** (1977), 437–71.
4. N. T. Shawagfeh and A. D. Alawneh, Submitted.
5. R. P. Kanwal and D. L. Sharma, *J. Elasticity* **6** (1976), 405–16.
6. N. T. Shawagfeh, *Bull. Fac. Sc. Alex.* (To appear).
7. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover Publications, New York, 1944.
8. T. Miloh, *SIAM J. appl. Math.* **26** (1974), 334–44.
9. G. B. Jeffery, *Proc. R. Soc.* **A102** (1922), 161–76.
10. A. I. Lur'e, *Three Dimensional Problems of the Theory of Elasttcity*, Interscience Publisher, New York, 1964.

SUGGESTIONS TO CONTRIBUTORS

The INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS is devoted primarily to original research in pure and applied mathematics.

Manuscripts should be typewritten, double-spaced with sufficient margins (including abstracts, references, etc.) on one side of durable white paper. The initial page should contain the titel followed by author's name and full mailing address. The text should include only as much as is needed to provide a background for the particular material covered. Manuscripts should be submitted in triplicate.

The author should provide a short abstract, in triplicate, not exceeding 250 words, summarizing the highlights of the principal findings covered in the paper and the scope of research.

References should be cited in the text by the arabic numbers in superior. List of references should be arranged in the arabic numbers, author's name, abbreviation of Journal, Volume number (Year) page number, as in the sample citation given below :

For Periodicals

1. R. H. Fox, *Fund. Math.* 34 (1947) 278.

For Books

2. H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin, (1973) p. 283.

Abbreviations for the titles of the periodicals should, in general, conform to the *World List of Scientific Periodicals.*

All mathematical expressions should be written clearly including the distinction between capital and small letters. Clear distinction between upper and lower cases of c,p,k,z,s, should be made while writing the expression in hand. Also distinguish between the letters such as 'Oh' and 'zero'; l(el) and 1 (one); v, V and ν (Greek nu); r and γ (Greek gamma); x, X and χ (Greek chi); k, K and κ (Greek kappa); Greek letter lambda (Λ) and symbol for vector product ($\wedge$); Greek letter epsilon (ϵ) and symbol for 'is an element of' ($\in$). The equation numbers are to be placed at the right-hand side of the page. The name of the Greek letter or symbol should be written in the margin the first time it is used. Superscripts and subscripts should be simple and should be placed accurately.

Line drawings should be made with India ink on white drawing paper or tracing paper. Letterings should be clear and large. Photographic prints should be glossy with strong contrast. All illustrations must be numbered consecutively in the order in which they are mentioned in the text and should be referred to as Fig. or Figs. Legends to figures should be typed on a separate sheet and attached at the end of the manuscript.

Tables should be typed separately from the text and placed at the end of the manuscript. Table headings should be short but clearly descriptive.

Proofs should be corrected immediately on receipt and returned to the Editor. If a large number of corrections are made in the proof, the author should pay towards composition charges. In case, the author desires to withdraw his paper, he should pay towards the composition charges, if the same is already done.

For each paper, the authors will receive 50 reprints free of cost. Order for extra reprints should be sent with corrected page proofs.

Manuscripts, in triplicate, should be submitted alongwith the declaration : "The manuscripts entitled .... by..............are submitted for publication only in the Indian Journal of Pure & Applied Mathematics and not elsewhere" to the Editor of Publications, *Indian Journal of Pure and Applied Mathematics*, Indian National Science Academy, Bahadur Shah Zafar Marg, New Delhi 110002 (India).

RN 19797/70 Regd. No. D-(C)-774

INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS

No. 8 August 1989 Volume 20

CONTENTS

Published and Printed by Dr O. N. Kaul, Executive Secretary, Indian National Science Academy, Bahadur Shah Zafar Marg, New Delhi 110002, at Leipzig Press, D-52, N.D.S.E. Part I New Delhi 110049. Ph. 622490